SUBSPACE IN LINEAR ALGEBRA: INVESTIGATING STUDENTS' CONCEPT IMAGES AND INTERACTIONS WITH THE FORMAL DEFINITION

Megan Wawro	George F. Sweeney	Jeffrey M. Rabin
San Diego State University	San Diego State University	Univ. of California San Diego
Univ. of California San Diego	Univ. of California San Diego	jrabin@math.ucsd.edu
megan.wawro@gmail.com	georgefsweeney@gmail.com	

Abstract

This paper reports on a study investigating students' ways of conceptualizing key ideas in linear algebra, with the particular results presented here focusing on student interactions with the notion of subspace. In interviews conducted with eight undergraduates, we found students' initial descriptions of subspace often varied substantially from the language of the concept's formal definition, which is very algebraic in nature. This is consistent with literature in other mathematical content domains that indicates that a learner's primary understanding of a concept is not necessarily informed by that concept's formal definition. We used the analytical tools of concept image and concept definition (Tall & Vinner, 1981) in order to highlight this distinction in student responses. Through grounded analysis, we identified recurring concept imagery that students provided for subspace, namely: geometric object, part of whole, and algebraic object. We also present results regarding the coordination between students' concept image and how they interpret the formal definition, situations in which students recognized a need for the formal definition, and qualities of subspace that students noted were consequences of the formal definition. Furthermore, we found that all students interviewed expressed, to some extent, the technically inaccurate "nested subspace" conception that $\mathbf{R}^k$ is a subspace of $\mathbf{R}^n$ for $k < n$. We conclude with a discussion of this and how it may be leveraged to inform teaching in a productive, student-centered manner.

Purpose and Background

A first course in linear algebra is commonly seen as a pivotal yet difficult mathematics course for university students. It is one of the most useful fields of mathematical study because of its unifying power within the discipline as well as its applicability to areas outside of pure mathematics (Dorier, 1995; Strang, 1988). According to Harel (1989), linear algebra is an important subject at the college level in that it: a) can be applied to many different content areas, such as engineering and statistics, because of its power to model real situations; and b) can be studied in its own right as "a mathematical abstraction which rests upon the pivotal ideas of the postulational approach and proof" (p. 139). In many universities, a first course in linear algebra follows immediately after a calculus series and, most likely, prior to an introduction to proof course. According to Carlson (1993), the majority of students' mathematical experiences up to that point have been primarily computational in nature. The content of linear algebra, however, can be highly abstract and formal, in stark contrast to students' previous computationally oriented coursework. This shift in the nature of the mathematical content being taught can be rather difficult for students to handle smoothly. Carlson posited that concepts are often taught without substantial connection to students' previously learned mathematical ideas, as well as without examples or applications. Thus, students struggle to connect familiar concepts to prematurely formalized, unfamiliar ones. Dorier (1995) contended that the serious difficulties students have in learning linear algebra are a result of its abstract and formal nature. Indeed, Robert and Robinet (1989) showed that prevalent student criticisms of linear algebra relate to the course's "use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics" (as cited in Dorier, Robert, Robinet, & Rogalski, 2000, p. 86).

As the quote implies, the mathematical activities central to linear algebra frequently involve mathematical ideas and definitions with which students have little experience, such as subspace or linear transformation. Investigating student thinking about fundamental mathematical ideas and definitions such as these is an integral part of basic research on the teaching and learning of linear algebra. Much of this research has focused on student difficulties and suggestions to alleviate these difficulties (e.g., Hillel, 2000; Sierpinska, 2000; Stewart & Thomas, 2008) or on students' creative and productive ways of reasoning about key concepts of linear algebra (e.g., Larson, 2010; Possani, Trigueros, Preciado, & Lozano, 2010; Wawro, 2009). A handful of studies explicitly reported on students' experiences with subspace. For instance, Harel and Kaput (1991) studied student responses to the following problem:

> Let V be a subspace of a vector space U, and let β be a vector in U but not in V.
> Is the set $V + \beta = \{v + \beta \mid v \text{ is a vector in } V\}$ a vector space?

Harel and Kaput categorized student responses into two main groups. Some students focused on the formal definition of vector space, checking each of the individual axioms for $V + \beta$, while others were able to view the vector space V as an entity while treating β as a shift operator that affected the existence of the zero vector. Students that used the latter approach, the authors argued, demonstrated a high level of mathematical sophistication in that they were able to reason through properties of vector spaces rather than check them algorithmically.

From the results of a large study regarding the teaching and learning of linear algebra in French universities, Dorier et al. (2000) reported many instances in which student difficulties seem to be due to a formalism obstacle, noting how "the lack of prior knowledge in logic and elementary set theory contribute to the production of errors in linear algebra itself" (p. 86). In an example regarding subspace, the authors found that when asked to give a basis for a particular subspace of $\mathbf{R}^4$, students could execute the correct steps and solve the relevant system of linear equations but would present their answer with vectors in $\mathbf{R}^2$ rather than $\mathbf{R}^4$, omitting the two components that were equal to zero. That is, students displayed a difficulty in choosing the correct symbolism to represent their solution, and a firmer grasp of set theory may have allowed students to see that their basis vectors, which were not even elements of $\mathbf{R}^4$, could not generate a subspace of $\mathbf{R}^4$.

In a study that explicitly addressed subspace, Britton and Henderson (2009) examined students' difficulties with the notion of closure. They analyzed how students responded to the following two questions:

1. Let $V = \{ t\langle 1,2,3\rangle \mid t \text{ is in } \mathbf{R} \}$. Show that V is a subspace of $\mathbf{R}^3$.
2. Let X be the set of functions f: $\mathbf{R} \rightarrow \mathbf{R}$ in $\mathbf{F}$ such that the graph $y = f(x)$ passes through the point (0, 0). Show that X is a subspace of $\mathbf{F}$ [$\mathbf{F}$ is the set of all such functions, without the restriction on the graph].

The authors found that students did, as expected, evidence difficulties with showing the closure of these sets under addition and scalar multiplication. Students tried to demonstrate these properties by choosing particular rather than general vectors. Students also struggled with "the formalism required to write a convincing proof, problems with logic and set theory, [and] the difficulty of moving from the abstract mode in which the definition is phrased to the algebraic mode in which the question is framed" (p. 969). The authors concluded that, in addition to the expected difficulties with proof and logic, many of the students' difficulties centered on interpreting the meaning of the various symbols.

As this body of research indicates, students do have difficulty with many of the mathematical ideas and activities central to the concept of subspace. In our study, however, we additionally found that students exhibited a variety of creative and powerful ways of conceptualizing subspace and interacting with its formal definition. The research presented in this paper grows out of a study that investigated the interaction and integration of students' conceptualizations of key ideas in linear algebra, namely subspace, linear independence, basis, and linear transformation. The particular results we present in this paper center around the notion of subspace. In interviews conducted with eight undergraduates, we asked students to describe how they think of subspaces of $\mathbf{R}^6$. Consider the formal definition of subspace:

> A nonempty subset V of $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$ if it is closed under addition and scalar multiplication. That is, if $\mathbf{x} + \mathbf{y}$ is in V whenever $\mathbf{x}$ and $\mathbf{y}$ are in V; also if $a\mathbf{x}$ is in V whenever $\mathbf{x}$ is in V and a is a real number.

The formal definition of subspace provides the machinery to determine whether a set is a subspace of a given space and, in some cases, may allow students to construct their own examples, but it does not seem to prompt any immediate imagery. Imagining higher-dimensional spaces can be difficult for any learner because our perceptual experience is limited to three spatial dimensions. The abstraction and the lack of conventional imagery may compel students to create meaning for the term as they interact with it in different problem situations. Investigating this—how students deal with making sense of this abstract, formal definition—was central to our study. In this paper, we utilize the constructs of concept image and concept definition (Tall & Vinner, 1981) to analyze these students' notion of subspace in linear algebra.

Theoretical Framework

The theoretical constructs of *concept image* and *concept definition* (Tall & Vinner, 1981) have proven to be a useful analytical tool for nearly three decades. Tall and Vinner defined *concept image* as "all the cognitive structure in the individual's mind that is associated with a given concept" (p. 151) and *concept definition* as "the form of words used to specify that concept" (p. 152). Vinner (1991) added that a person's concept image can be a visual representation, as well as a collection of impressions or experiences associated with that concept. A concept image is cumulative, changes over time, and is not simply a static item stored in memory. Particular aspects of it may be activated in response to some questions or problem situations, while other aspects are not. Because of this, Tall and Vinner define an *evoked concept image* as the "portion of a person's concept image that is activated at a particular time" (p. 152). Furthermore, any concept image may have aspects that differ from the concept's formal definition, and these distinctions may not be coherent across all problems and situations. For instance, students may think of vectors as arrows anchored at the origin, as points in a coordinate system, or as floating directed line segments. While any one of these concept images of vector may be useful in a specific situation, combining them within the same context could prove problematic for inexperienced linear algebra learners. This situation—when one aspect of a student's concept image conflicts with another aspect of that same concept image or definition—is referred to as a *potential conflict factor* (p. 153). We return to this notion of potential conflict factors in the discussion section, in which we offer possible positive implications for teaching from evoking these conflict factors within the context of coordinating concept images of subspace with that concept's formal definition.

Zandieh and Rasmussen (2010) identify two major strands of research regarding concept image and concept definition. The first focuses on cataloging learners' concept images about a particular mathematical idea (e.g., Artigue, 1992; Edwards, 1999; Zandieh, 2000). The second strand of research deals with cataloging difficulties that arise when students reason from their concept image even though they "know" the concept definition (e.g., Bingolbali & Monaghan, 2008; Edwards & Ward, 2008; Vinner & Dreyfus, 1989). For example, Vinner (1991) discussed three possible outcomes when a student is introduced to a new concept definition for which he already has some sort of concept image. The student could: (a) change his concept image in order to encompass the new definition, (b) keep his concept image the same and ignore or forget the new definition, or (c) use both his concept image and the new definition but without integration of the two. In the case of subspace in linear algebra, our study contributes to both strands of this research. In the results section, we first present our findings regarding various student concept images for subspace. In the remainder of the section we discuss various aspects of how students interacted with the formal definition of subspace and ways in which this interaction was consistent with or in conflict with their previously described concept images of subspace.

There also exists a fair amount of research regarding how one could use the concept image/concept definition framework as a tool to influence pedagogy at the undergraduate level. Tall and Vinner (1981) argued that, in the area of real analysis, weak understandings of a formal definition, coupled with a strong concept image, can make the process of formal proof difficult for students. Bingolbali and Monaghan (2008) made explicit the assumption that students bring a variety of experiences and ideas to their work on mathematical tasks. The authors studied student concept images of the derivative, arguing that departmental affiliation (mechanical engineering or mathematics) affected the development of concept images and should not be ignored. Finally, Zandieh and Rasmussen (2010) make use of the concept image/concept definition framing to extend the notion of defining as a mathematical activity. The authors examine how students actually use and create concept images and concept definitions in an undergraduate non-Euclidean geometry course, in order to offer a structured way to aid practitioners in planning for and analyzing the ways in which students define as they transition from intuitive ways of reasoning to more conventional, formal ways of reasoning in mathematics. In the discussion section, we offer implications for teaching that involve productively making use of students' intuitive (but technically incorrect) notion that $\mathbf{R}^k$ is a subspace of $\mathbf{R}^n$ for $k < n$. We posit this as an example of a pedagogical opportunity to foster students' creation of a concept image of isomorphism that may be further refined to the formal definition of isomorphism.

Setting and Methods

The subjects for this study were first-year university students enrolled in a one-year honors calculus course at a large public university in the southwestern United States. All students had completed Advanced Placement (AP) Calculus BC (a full-year college-level course in single variable calculus) in high school and had earned a 5 (the highest possible score) on the national AP (BC) examination to qualify for two quarters of calculus credit at the university. The first quarter of the honors course presented linear algebra as providing the formal structure for extending geometric ideas from $\mathbf{R}^2$ and $\mathbf{R}^3$ to $\mathbf{R}^n$, so as to support the subsequent quarters on multivariable calculus in the full generality of n dimensions. Accordingly, the setting was limited to $\mathbf{R}^n$; abstract vector spaces were not discussed. The notion of subspace was introduced in the course via the formal definition given above, motivated by the intention to identify “flat” subsets of $\mathbf{R}^n$ that might serve as “tangent spaces” to curved graphs in calculus. Examples included, initially, lines and planes, and, later, the span of a list of vectors in $\mathbf{R}^n$. Connecting formal definitions with geometric examples was an explicit goal of the course. In the two subsequent quarters of the course, tangent spaces were indeed defined using subspaces, and the derivative was defined as a linear transformation best approximating a function.

Approximately three weeks after the conclusion of the linear algebra quarter, eight students participated in hour-long, individual, semi-structured interviews (Bernard, 1988). Of the 24 that volunteered (total enrollment was 33) to be interviewed, the instructor selected as a representative sample four male and four female students with course grades ranging from A to C. Data sources consisted of video recordings of the interviews, students’ written work, and researcher field notes. All interviews were transcribed completely. As stated, the broad goals of the interview were to investigate students’ conceptualizations of foundational ideas in linear algebra as well as to investigate how students coordinate their developed imagery with the formal definition for these concepts. We focus on results pertaining to subspace in this paper. In line with these goals, the following question served as a broad frame for inquiring into student thinking regarding subspace:

How do you think about what a subspace of $\mathbf{R}^6$ is?

To further elucidate their conceptualizations, we asked students to provide examples of subspaces of $\mathbf{R}^6$, state why those are viable examples, and whether there is anything necessary for something to qualify as a subspace. In addition, we asked students the following questions and to explain their answers.

a. Consider the vectors <1, 2, 3, 5>, <2, 5, -3, 1>, and <2, 4, 5, -7>. Do these vectors form a subspace?
b. On this sheet of paper is your textbook’s definition for subspace. Read it aloud and explain how you think about it.
c. How does the textbook definition relate to your previous description of how you think of subspace?

In follow-up question (a), one may notice that to say three vectors “form” a subspace is mathematically imprecise. It would be a more proper question to ask whether these vectors form a basis for a subspace, or to ask for an alternative description of the subspace these vectors span. This question was intentionally vague in order to investigate how students thought about how subspaces are “built.” This vagueness also allowed us to see how students interacted with the vectors as elements of a vector space. For instance, deciding whether these vectors are in $\mathbf{R}^3$ (because there are three vectors) or $\mathbf{R}^4$ (because each vector has four components) was not trivial for some.

In designing the interview, we intentionally phrased our questions in such a way as to foster an environment that would be conducive to students offering their own ways of thinking. For instance, we asked students “how they think about what a subspace is” rather than “what a subspace is.” The phrasing was chosen to minimize the likelihood that students would conclude that we were looking for a particular “correct” answer, as well as to give us the greatest opportunity to elicit students’ actual concept images. Furthermore, we created prompts that asked students about their thinking rather than giving them problems to solve. By doing so, students were given the opportunity to describe their visualization of ideas and to reflect on their own knowledge and thought processes rather than get “bogged down” in long calculations or numerical errors. We were able to retrospectively examine the data to see what questions prompted students to use the formal definition as opposed to only their concept images, as well as to see whether students spontaneously referred to definitions and for what purposes they did so. For example, one student, Shelley[1], shared how she personally sees the utility of both her concept image and the formal definition of subspace: “If I had to prove a subspace I would use this [the definition], but when I just think about it, I just think about it as a space.” Thus, the design of our interviews allowed us to not only examine students’ concept images but also how students connected these images to and found use for the formal definition of subspace. As stated above, an evoked concept image is dependent on the question or problem presented to the student. Thus our findings may complement those of other studies in which students were given more structured problems to solve.

For the purposes of analysis, we used grounded theory (Strauss & Corbin, 1990). During our first round of analysis, we independently created open codes to categorize the kinds of language that students used when discussing the concept of subspace and interacting with the definition. We then used axial coding to independently create and flesh out the concept image categories for subspace that emerged from the data. Following each round of analysis, we met to discuss our notes, summaries, and codes. Transcripts were re-read, video data re-examined, and codes and inferences were discussed to ensure investigator triangulation (Denzin, 1978). Once we agreed on the existence of three main categories that captured the various ways in which students described subspace, we began the process of selective coding (Strauss & Corbin, 1990), during which we focused our further investigation on three specific topics: (a) coordination between students’ language to describe subspace and to interpret the formal definition, (b) instances of students spontaneously invoking a use for the formal definition of subspace, and (c) situations in which students generated consequences of the definition. Finally, we returned to the data in order to investigate instances in which students explicitly referred to $\mathbf{R}^k$ as a subspace of $\mathbf{R}^n$. Each of these investigations is discussed in the results section, with the last serving as the foundation for a discussion regarding positive implications for teaching.

Results

From our analysis, we found that the eight students interviewed used a variety of language to describe their personal notions of subspace. From this variety, we identified three main categories for students’ concept images of subspace. We also present results regarding the coordination between students’ concept image and how they interpret the formal definition, situations in which students perceived a need for the formal definition, and properties of subspace that students noted were consequences of the formal definition.

Categorization of Students’ Concept Images of Subspace

The three categories of concept image that emerged from our data were: Geometric Object (e.g., subspace is a *plane* in a space), Part of a Whole (e.g., subspace is *contained within* a space), and Algebraic Object (e.g., subspace is a *collection of vectors*). We define and elaborate on each of these categories and provide examples of each.

---

[1] All names are pseudonyms.

Subspace as Geometric Object. Seven of the eight students referred to subspace as some familiar object in geometry, such as a plane, line, cube, or sphere. Many of the students used multiple familiar geometric terms in order to describe how they think of subspaces. For example:

Interviewer: How do you think about what a subspace of $\mathbf{R}^6$ is?
Henry: I typically just think about a line, a plane, and the 3-D space. Because obviously you can't imagine $\mathbf{R}^4$, $\mathbf{R}^5$, $\mathbf{R}^6$. So I just kind of imagine those spaces.

Throughout the interview, Henry repeatedly mentioned that it is not possible to imagine anything of a higher dimension than $\mathbf{R}^3$. Indeed, Henry noted that he conceptualizes $\mathbf{R}^6$ as having six orthogonal coordinate axes, later saying, "If you ask me something in ... more than $\mathbf{R}^3$, I'll have to think about it in $\mathbf{R}^2$ and $\mathbf{R}^3$ and sort of expand it, ... and then hopefully it'll be the same in $\mathbf{R}^6$." Henry's appeal to geometry to describe subspaces of $\mathbf{R}^6$ demonstrates the power of this type of imagery for him. In fact, Henry uses a perceptually impossible image, "six orthogonal coordinate axes," in order to describe $\mathbf{R}^6$. After Henry stated that he has to think in $\mathbf{R}^2$ and $\mathbf{R}^3$, the interviewer prompted Henry to say more about this:

Interviewer: What about if I asked about subspaces of $\mathbf{R}^3$, can you talk about that?
Henry: I just think about planes and lines.

This geometric imagery persisted throughout Henry's interview. The main interview prompt for subspace (asking how students think about a subspace of $\mathbf{R}^6$) and its follow-ups were directly focused on eliciting student imagery. Henry's responses to other questions in the protocol—questions that did not explicitly ask about his imagery—were consistent with this geometric language.

Another student, Nancy, provided responses that also exemplified the Geometric Object view of subspace. When asked how she thought about subspaces of $\mathbf{R}^6$, Nancy's first response was, "I guess sort of any plane that lies in, somewhere in $\mathbf{R}^6$, which you can't really visualize ... But I guess a plane or some other multi-dimensional space or object that would fit into whatever the $\mathbf{R}^6$ would look like." Her first description relied on a geometric conceptualization of space, even though "you can't really visualize" $\mathbf{R}^6$. Within the first five minutes of the interview, Nancy referred to subspaces as planes six more times. One of these occurred in the following example, where the interviewer asked Nancy to switch from thinking about subspaces of $\mathbf{R}^6$ to those of $\mathbf{R}^3$.

Interviewer: ...How about $\mathbf{R}^3$?
Nancy: I don't know, I sort of visualize it as almost a sphere, but with sort of fuzzy edges, I guess.
Interviewer: Can you tell me what a subspace of $\mathbf{R}^3$ would look like?
Nancy: I mean $\mathbf{R}^2$ would also be, so a plane through $\mathbf{R}^3$.
Interviewer: So $\mathbf{R}^2$ and a plane through $\mathbf{R}^3$, those things are the same thing?
Nancy: Yeah.

In this example, Nancy was explicit in her thinking that $\mathbf{R}^2$ is a subspace of $\mathbf{R}^{3,}$ rather than expressing the formally correct notion that two-dimensional subspaces of $\mathbf{R}^3$ are isomorphic to $\mathbf{R}^2$. Because of the frequency with which Nancy referred to subspace as a plane, we claim that her most prevalent evoked concept image for the subspace was that of Geometric Object.

In summary, student utterances that fit into this category include those that featured familiar geometric objects such as lines, planes, spheres, and cubes. Students' imagery focused primarily on reasoning in $\mathbf{R}^2$ and $\mathbf{R}^3$, and students frequently discussed difficulties in imagining $\mathbf{R}^6$—as evidenced by Henry's earlier utterances. However, students did discuss how working in $\mathbf{R}^2$ and $\mathbf{R}^3$ helped them to visualize what a subspace of $\mathbf{R}^6$ might be. Students' responses do not always reflect only a single concept image code. Instead, different aspects of any given response may fit with multiple codes. In addition to Subject as Geometric Object, Nancy's description utilized Part of a Whole imagery (the subspace would *fit into* $\mathbf{R}^6$). We describe this second category now.

Subspace as Part of a Whole. Six of the eight students described subspace in light of a relationship to some larger, encompassing "parent space" (our term). The terms students used to reflect this relationship were *contained within, part of, smaller than,* and *inside of.* What separates this category from the previous one is the lack of explicit reference to a particular geometric shape. For instance, when Shelley was asked to describe how she thought about a subspace of $\mathbf{R}^6$, she replied:

Shelley: A subspace. I just think of it as a part of being in $\mathbf{R}^6$. You have different dimensions in that, but it can be all of $\mathbf{R}^6$ as well. I don't think of it as a space, exactly, I visualize 3-D but not exactly like, yeah.
Interviewer: You don't visualize it, you don't visualize $\mathbf{R}^6$, is that what you're saying?
Shelley: Yeah, I can, but I just, the vision I get is $\mathbf{R}^3$ I guess, but I just think of a smaller space inside that.

Note that Shelley was explicit in describing how she does not necessarily see a specific space, but rather that she thinks of a subspace of $\mathbf{R}^6$ as a *part of being in* and a *space inside* $\mathbf{R}^6$. Because of the frequency with which Shelley referred to a subspace in relation to some other parent space but without a strong object-like description, we claim that her main evoked concept image was Subspace as Part of a Whole. Indeed, throughout the entire interview Shelley never described subspace as a geometric object. As another example, when asked how she would explain subspace to her roommate, her answer was similar.

Shelley: Um, well, I guess that's the only way I can explain it, because I just think of it as a space that's smaller than or all of the given $\mathbf{R}^n$.

Answers of this type were frequent and arose as a result of various prompts. Possible explanations for this imagery might lie in the language of the definition or the traditional connotations of the prefix "sub."

Subspace as Algebraic Object. Four of the eight students referred to subspace in a way that we categorized as Subspace as Algebraic Object. In broad terms, descriptions that fit into this category are set-theoretic rather than geometric. Students used terminology such as *element of* rather than *part of*, language more consistent with an abstract view of subspace. Students may have described subspace as a set of vectors or a matrix satisfying some condition, by a list of components, or by a system of equations it satisfies. Such a view is *algebraic* in the sense of Hillel's "algebraic language of linear algebra" (Hillel, 2000). For instance, when asked how he thinks of a subspace of $\mathbf{R}^6$, Mark responded by first describing how he imagines $\mathbf{R}^6$ and then how he envisions subspaces of $\mathbf{R}^6$.

Mark: I think of, I mean originally when I entered the class, I tried to think of everything in terms of 3-D, and then of course when we got to $\mathbf{R}^6$, yeah, I started re-imagining it. I don't know, I still like to think of it spatially as just, you see some of those complex boxes that mathematicians draw every now and then to try to illustrate. So I tried to think of it like that. But after linear algebra, it's really just a matrix with 6 entries for me, I really couldn't work with anything else. So I really, when I see $\mathbf{R}^6$, it kind of now has become just an algebraic term for me, it's a subspace in this case would be any dimensional or geometric thing that could fit into that matrix with 5... With 5 entries, 5, if it's $\mathbf{R}^6$, it fits anything with **R** being less than 6, it'll fit any object or any sequence with values that are, and then of course the largest subspace will be $\mathbf{R}^6$ itself.

When asked how he thinks of subspaces in $\mathbf{R}^6$, Mark said that he used to find it helpful to imagine subspaces spatially in terms of three dimensions, but that he no longer does so. Rather, he now sees subspaces as algebraic objects. He also said that the pertinent qualities of subspace are, for him, based on characteristics of vectors and matrices. He mentioned needing to know how many components each have, etc. This is in contrast to the two previous categories that were based on imagery that could be used independently of the content domain of linear algebra if one so chose. The imagery in the first two categories was heavily visual, whereas here—although we do see visual imagery to some extent—it is mainly based on mathematical objects such as vector and matrix.

We find it interesting that only four of the eight students used algebraic terminology to describe how they thought about subspaces, given that the formal definition of subspace is very algebraic and set-theoretic in nature. We wondered, then, if students hold their concept image and concept definition in isolation from each other. Do students use consistent language when describing how they think of subspace and when interpreting the formal definition? Because of the definition's strong set-theoretic wording, did more students utilize algebraic language to interpret the formal definition? These questions are the focus of the following sections.

Coordination between Concept Image and Formal Definition

During the interviews, students were first asked to describe how they thought about subspaces of $\mathbf{R}^6$. The interviewer then presented them with their textbook's definition of subspace, asking them to first interpret it and then to relate that interpretation to their previous statements. Invariably, students interpreted the concept definition in ways that were consistent with their earlier image of subspace. We found that each students' language continued to be consistent with the Algebraic Object, Geometric Object or Part-of-a-Whole imagery they had used before introduction of the definition. Students also elaborated on their previously evoked concept images with imagery that was not used for answering earlier questions. Regarding the first and third concept image categories—Geometric Object and Algebraic Object—students used language consistent with these categories to interpret the definition only if their prior description of subspace had used the same language. We found this somewhat surprising given the inherent algebraic formulation of the formal definition itself, which is restated below:

> A nonempty subset V of $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$ if it is closed under addition and scalar multiplication. That is, if $\mathbf{x} + \mathbf{y}$ is in V whenever $\mathbf{x}$ and $\mathbf{y}$ are in V; also if $a\mathbf{x}$ is in V whenever $\mathbf{x}$ is in V and a is a real number.

A second result concerns the remaining category—Subspace as Part of a Whole. Every student used language consistent with this concept image category when interpreting the formal definition. This may be because the formal definition itself uses language consistent with Part of a Whole imagery. The definition includes closure under addition and scalar multiplication, and the word "in" is prominently used throughout.

As stated above, we found it surprising that more students did not utilize Algebraic Object imagery when interpreting the formal definition. Students who used geometric imagery before introduction to the definition consistently connected their geometric imagery to their interpretation of the definition. As an illustrative case, consider Henry's explanation of how he understands closure under addition and scalar multiplication geometrically. Italics were added to highlight Henry's repeated geometric language.

Henry: Basically I think of a *plane* again, it's the *x-y* plane. And then I think just pick two vectors, whatever vectors, don't have to go orthogonal, whatever length. And then if you add them. And then I'll *imagine the parallelogram rule*, you add them, and then if it's *in the plane*, and then it's a subspace. Then it's closed under addition. Then I'll think about multiplication. I'll just pick a vector and then I'll *go forever in one direction* [extends an arm out and points towards the distance], you multiply by a whole bunch of numbers in *one direction*. And then multiply by some negative number, and you'd go *another direction*. And that should be in the subset.

In this explanation, Henry used a plane as a generic example of subspace. He described both vector operations, and the meaning of closure, in geometric terms. First, Henry discussed imagining closure under addition as creating a parallelogram on a plane. Henry then illustrated how scalar multiplication can be thought of going forever in both the positive and negative directions. This description of how the definition can be imagined geometrically is consistent with the concept image of subspace that Henry exhibited before he was given the definition.

Some students did use algebraic language in using the definition to relate to other interview prompts, and we see this in a variety of ways. We illustrate this variation with two examples. In the first, Amy tried to use the definition to determine whether a set of three given vectors in $\mathbf{R}^4$ (see follow-up question (a) under Methods) form a subspace:

Amy: I'm just looking to see if I can match this definition with what I'm doing here, because there are two things here [in the definition] and there are three vectors here [in the problem]. And I was just seeing if I could correlate with the [inaud].

Interviewer: How are you trying to do that?

Amy: I'm just trying to put the vectors, name each of the vectors, so it's **x** and **y**. And I was thinking of scalars, and then I was thinking of adding them, and if they would still be in the subspace? Then do I need to, is there, do I need to say it's a subspace of $\mathbf{R}^n$, is this what it's asking, is this a subspace of $\mathbf{R}^n$ or is this a subspace of anything, or?

When given the definition, Amy literally tried to match up the three given vectors with the two vectors appearing in the definition. The mismatch caused her to struggle in interpreting the definition. Amy's literal interpretation of the definition algebraically was consistent with her evoked concept image, as she frequently discussed subspace in terms of dimensions and adding additional basis vectors. When dealing with the definition, Amy saw the **x** and **y** not as arbitrary vectors within the space, but as specific vectors that needed to be tested in a procedural way.

Mark's use of the definition also seemed consistent with a Subspace as Algebraic Object concept image, but in a different way. Instead of focusing on the presence of two vectors in the formal definition, he interpreted closure under addition and scalar multiplication as meaning all linear combinations of the vectors. The following excerpt is again from follow-up question (a).

Interviewer: How would you describe that subspace then?

Mark: I feel it's just points, actually, it's just the point (1,2,3,5). Under this, it's not all of what, it's not all possible additions or multiplications, or sums and products of these matrices, it's just the points, the three points. Unless we define the subset as all linear combinations of the two. So it's just three points in the $\mathbf{R}^4$, just existing in space around, in $\mathbf{R}^6$.

Mark noticed that the set that he was given was not all linear combinations of the vectors, but only three points of $\mathbf{R}^4$. After the interviewer suggested that they change the question's wording, Mark's response combined algebraic and geometric imagery:

Interviewer: What if I change my question a little, what if I said do these vectors form a basis for a subspace, would that change your?

Mark: Yes, that would change it, because then it would be all combinations of those, and all multiples of those vectors [points to the set of three vectors]. Then it would be in fact a, it would be like a geometric figure kind of. I don't really know how I, I can't really visualize it in $\mathbf{R}^4$, but it would be kind of, I think the best metaphor I can think of would be a small bowl inside a cube, kind of like that. It would be a geometric shape, or like a plane through 3-dimensional space sort of deal. Because then this [points to the set of three vectors again] would be, this would go, would be a form in $\mathbf{R}^4$ that actually, I guess this would span if it's linearly independent of one another, it would span $\mathbf{R}^4$. So you'd have like an $\mathbf{R}^4$ plane going through $\mathbf{R}^6$ subspace.

A basis implied for Mark that the set includes all linear combinations of the vectors, allowing for a different shape, specifically "a small bowl inside of a cube." The algebraic imagery of all of the linear combinations of the given vectors led him to draw the conclusion that the set is no longer just three points, but instead is a figure, which he characterized as "a geometric shape…a plane through three-dimensional space sort of deal." When attempting to determine if three given vectors form a subspace, Mark interpreted the definition using the algebraic imagery of linear combinations, which he connected to his geometric imagery.

Mark's responses illustrate the powerful role that definitions can play in solidifying and developing a student's concept image. Other students' understanding of the definition affected their concept image in a variety of ways. In Amy's case, her literal interpretation of the definition displayed her reliance upon algebraic imagery and computation to make sense of subspace. Henry showed how he created a geometric understanding of the definition by relating it to his previously evoked concept images, whereas Mark showed how the concept definition can bring together disparate aspects of a student's imagery into a more cohesive whole. Mark's understanding of the definition combines his algebraic and geometric reasoning through the mechanism of closure under addition and scalar multiplication.

Necessity for and Consequences of the Formal Definition

Knowing when a definition would be a valuable resource is important for learning mathematics. We analyzed when the students first requested, mentioned, or produced the definition of subspace as indicators of the purposes definitions can serve for them. Tall and Vinner (1981) noted that students relied upon their concept images rather than the concept definition to solve problems and make sense of mathematics and further claimed that this reliance could be a hindrance to the students. In our analysis, we found that students frequently saw the use and value of the definition of subspace and knew when they would use it in order to make sense of the mathematics.

Seven of the eight students invoked the definition prior to being given the textbook definition, some doing so multiple times. Four of the seven mentioned or gave the definition during the task of determining whether the given set of three vectors in $\mathbf{R}^4$ form a subspace. Amy went so far as to refuse to answer the question until the definition was provided, stating, "Would you tell me the definition of a subspace? Because that would help!" The other three students were not as insistent, but they also requested the definition in order to solidify their responses. Another student, Shelley, gave the definition to support her claim that a certain set of vectors was *not* a subspace. These responses indicate that students see the need for the definition to affirm, mathematically, that an object or set of objects is in fact a subspace.

In other examples of seeing the need for a formal definition, students referred to it in order to elaborate or explain features of their concept image. Two consequences of the concept definition that were cited by students were that a subspace must contain the zero vector and must be "flat." Both had been emphasized in the class. After Nancy provided her initial thoughts about subspaces of $\mathbf{R}^6$, the following exchange occurred:

Interviewer: What is necessary for something to be a subspace?

Nancy: I don't remember the definition of a subspace off the top of my head...Usually I think of it as like some sort of flat thing that, I remember it has to go through the origin. But usually a flat, either a plane or a line or something that's two-dimensional.

After being provided with the definition and asked how it related to her previously given image of subspace, she made the point more explicitly:

Nancy: It's sort of related to the picture I had in my head about some sort of flat object, or a plane, or a line or whatever. Because a subspace has to be flat or else it won't be closed under addition.

For Nancy, the intuitive concept of flatness is captured by the definition. Furthermore, Nancy's statement evidences that the definition and the image are in fact connected for her in meaningful ways. As noted above, this connection was made in the course. Despite Nancy's example of something two-dimensional, "flatness" as used in the course is meaningful in any dimension, being linked to closure as she states later. The intuition is that any two linearly independent vectors in a subspace determine a plane containing them, their sum, and indeed all their linear combinations. Closure means that the subspace must contain this plane, rather than bending away from it as a curved surface would, so subspaces as defined are intuitively "flat."

Flatness was not the only consequence of the definition that students mentioned before being given the definition. Containing the zero vector appears to be part of the concept image for many students, as evidenced by the fact that this vector was explicitly mentioned by the majority of students themselves, without prompting from the interviewer. This had also been emphasized in the course, and two students specifically cited it as a consequence of the definition.

Interviewer: Could any plane be a subspace of $\mathbf{R}^3$?
Henry: It has to pass through the origin.
Interviewer: Why so?
Henry: Because that's the definition of a subspace...It's not a definition, but it's a little thing that comes along with the definition. It has to be closed under vector addition, and closed under scalar multiplication, and if you're both of those, then you pass through origin.

Henry first gave the definition after being asked whether any plane could be a subspace. This implies that Henry saw the definition as being important for specifying what is necessary for a plane or other object to be a subspace. More specifically, he invoked the definition to explain why the zero vector must be part of the subspace. He first suggested that this *is* the definition, but then clarified that it is a *consequence* of the definition. Henry parsed for himself an aspect of the relationship between his concept image and the concept definition by explicating the relationship between a subspace passing through the origin and being closed under scalar multiplication and addition. Note that Henry said a subspace must "pass through the origin," as opposed to "must contain the zero vector." This distinction is additional evidence for his geometric object, rather than algebraic object, image of subspace. Two other students, Shelley and Nancy, also pointed out that any subspace must contain the origin. In addition, they both gave a proof, by showing that otherwise the subspace would not be closed under scalar multiplication (where the scalar is zero).

The Nested Subspace Conception

Tall and Vinner (1981) pointed out that a person's concept image may contain elements that conflict with the formal definition. These *potential conflict factors* may obstruct students' learning of the formal theory built around the concept in question. One example of such a conflict between concept image and concept definition is what we refer to as the notion of "nested subspaces," which was exhibited to some extent by all students interviewed. This is the view that the spaces $\mathbf{R}^n$ are nested, in the sense that $\mathbf{R}^2$ is a subspace of $\mathbf{R}^3$, which is a subspace of $\mathbf{R}^4$, and so forth. It often includes the view that *any* 2-dimensional subspace of $\mathbf{R}^n$ "is" $\mathbf{R}^2$. Although students' tendency towards this conception may be known by some linear algebra instructors or mathematics education researchers, we were unable to find specific documentation of this in the research literature. Thus, we describe this example of a potential conflict factor and suggest how it may be leveraged towards useful pedagogical opportunities.

Adhering strictly to the definition of these spaces, the view that $\mathbf{R}^k$ is a subspace of $\mathbf{R}^n$ for $k < n$ is false. For example, $\mathbf{R}^2$ is the set of ordered pairs of real numbers (x, y), while $\mathbf{R}^3$ is the set of ordered triples (x, y, z), so the former is not even a subset of the latter. In terms of desirable concept images, however, the *x-y* plane certainly is a subset and a two-dimensional subspace of three-dimensional space equipped with a Cartesian coordinate system. All planar subspaces of $\mathbf{R}^3$ can be obtained by rotating the *x-y* plane, and one often views rotations as not changing the identity of a geometric object. The dimension of a subspace is its most important characteristic and gives the number of parameters needed to determine a vector in that subspace. Because of this, we *want* students' concept images to include the idea that subspaces of the same dimension are interchangeable with regard to their internal structure, even though their locations within the parent space may differ.

Students expressed these beliefs about "nested subspaces" in our interviews in several ways. For instance, Amy offered $\mathbf{R}^2$ and $\mathbf{R}^3$ as examples of subspaces of $\mathbf{R}^6$ during her initial response to the first question of the interview.

Interviewer:	The first question we have is, how do you think about what a subspace of $\mathbf{R}^6$ is?
Amy:	Do you want me to write it down?
Interviewer:	You can write it down, you can describe it to me, anything that you feel comfortable doing.
Amy:	I'll describe it. I think of it as something that makes up a part, that makes up $\mathbf{R}^6$, like $\mathbf{R}^2$, $\mathbf{R}^3$, the stuff below $\mathbf{R}^6$, that's what I think.
Interviewer:	So $\mathbf{R}^2$ and $\mathbf{R}^3$ and $\mathbf{R}^4$, those things make up $\mathbf{R}^6$?
Amy:	Yeah, they're part of it, they're one of the building blocks or components.

While Amy's language is consistent with the Subspace as Part of a Whole concept image category, Nancy mentioned $\mathbf{R}^2$ as a subspace of $\mathbf{R}^3$ with language consistent with the Subspace as Geometric Object imagery. Recall the following transcript, repeated here for convenience:

Interviewer:	Can you tell me what a subspace of $\mathbf{R}^3$ would look like?
Nancy:	I mean $\mathbf{R}^2$ would also be, so a plane through $\mathbf{R}^3$.
Interviewer:	So $\mathbf{R}^2$ and a plane through $\mathbf{R}^3$, those things are the same thing?
Nancy:	Yeah.

Students did not just assert this view when asked about subspaces directly. Rather, students' explanations were consistent with this view in a variety of contexts throughout the interview. Some seemed to use $\mathbf{R}^k$ as a way of referring to any *k*-dimensional subspace of $\mathbf{R}^n$, for example stating that three linearly independent vectors in $\mathbf{R}^6$ would span a subspace which would be $\mathbf{R}^3$. Some students used the nested subspace conception in an explanatory role with ideas such as span and basis. In explaining why vectors in $\mathbf{R}^2$ cannot span $\mathbf{R}^3$, Nancy spoke as if R2 was realized as a plane in R3: "I guess if you have $\mathbf{R}^2$ as a flat plane and then $\mathbf{R}^3$, they can't describe anything that's going to be above that plane or outside of that plane...they can only describe points with two variables." Ted, when asked why a basis for $\mathbf{R}^n$ must contain *n* vectors, replied that if there were only *n*-1 vectors, they would lie in $\mathbf{R}^{n-1}$ instead. Mark (see below) realized that three vectors in $\mathbf{R}^4$ cannot literally span $\mathbf{R}^3$ because the vectors have four components, but he still asserted that $\mathbf{R}^4$ is nested inside $\mathbf{R}^5$ and so forth.

We were unable to identify any obvious sources for these views in the course content itself. One possible explanation is the proposed difficulty students have between moving between the different modes of description and representation of vectors, namely the geometric mode in which vectors are directed line segments emanating from a common point, and the algebraic mode where vectors are *n*-tuples of real numbers (Hillel, 2000). The geometric representation commonly used for $\mathbf{R}^2$ is a plane, and two-dimensional subspaces in $\mathbf{R}^k$ (with $k > 2$) are represented by planes as well. It is possible that students infer because the geometric representations for the two mathematical objects are the same that $\mathbf{R}^2$ must be a subspace of $\mathbf{R}^k$. Hillel posits that students' familiarity with the geometric mode may hinder the learning of more abstract aspects of linear algebra. Rather than focus on students' comfort with geometric images for subspace solely as a limitation, we choose to highlight how the notion of "sameness" may be leveraged as a way to necessitate the notion of isomorphism.

Mathematicians are familiar with the subtleties of uncritically regarding two mathematical objects as the same and deal with this conflict by means of the concept of *isomorphism*, where an isomorphism of vector spaces is an invertible linear mapping between them. Such a mapping between $\mathbf{R}^2$ and the x-y plane of $\mathbf{R}^3$, for instance, could be given by sending (x, y) to $(x, y, 0)$. Of course, there is no reason the extra zero must appear in the third position—the y-z or x-z plane would serve as well. Any two-dimensional subspace of any $\mathbf{R}^n$ is isomorphic to $\mathbf{R}^2$, but there is no standard isomorphism between them: a choice of basis $\{\mathbf{v},\mathbf{w}\}$ for the subspace of $\mathbf{R}^n$ must be made in order to identify (x, y) with $x\mathbf{v} + y\mathbf{w}$. An isomorphism allows two subspaces to be regarded as the same by providing the extra information of how their points are to be matched with each other. This essentially requires the specification of a coordinate system in each subspace; points having the same coordinates are then matched. There is no "canonical" or preferred way of doing this, but infinitely many possible choices.

In our data, we found many instances in which the students' explanations were consistent with the nested subspace notion. Rather than focus on this as a misconception, however, we choose to view the students as in the process of constructing an early concept image of isomorphism. We emphasize "in the process of" and "early" in the previous statement. We, by no means, contend that students fully grasp the subtleties of the notion of isomorphism. Rather, we contend that building from students' current ways of reasoning is a powerful and effective pedagogical approach. In the development of their images of subspace, we have evidence that students want to regard some subspaces as "the same." Instructors should be aware of the potential confusion this view of "sameness" may cause, as well as how it may provide an opportunity to necessitate the tool of isomorphism. The Necessity Principle (Harel, 2000) states that in order for students to learn, they must see the intellectual necessity for the concepts and procedures being taught. Hence, identification and elaboration of this particular problem can help teachers necessitate the concept of isomorphism. Teachers who are aware of this developing issue can pose questions and help students analyze and refine their notion of "the same" toward that of "isomorphic." In this way, the conflict between common concept images and the formal definition of subspace provides a pedagogical opportunity to build from students' informal concept images of isomorphism in order to develop a concept definition of isomorphism.

Conclusion

Work in understanding students' concept images in relation to concept definitions has frequently focused on how inconsistencies between the concept image and the concept definition can cause problems for students in developing mathematically correct understanding (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). Rather than cataloguing the difficulties associated with students' use of the concept image for subspace, we examined productive ways the formal definition was integrated into their concept image. The goal of this study was to investigate the ways that students conceptualize the fundamental idea of subspace and how those conceptualizations are connected to the idea's formal definition. Subspace was an appropriate concept to explore because of the contrast between the abstract, algebraic language used in the formal definition and the geometric imagery provided in students' explanations. Our analysis showed that students interpreted the formal definition in terms consistent with their rich concept images. They could connect specific aspects of their images, such as flatness and inclusion of the zero vector, to the algebraic closure specified by the definition. They realized the usefulness of the definition for identifying examples and nonexamples, as well as for justifying properties of subspaces. Thus, this study demonstrates how, in linear algebra, definitions can be an important tool in developing overall intuitions about mathematical concepts. We note that our population of honors students likely corresponds to the top tier of a typical introductory linear algebra class, and their responses may constitute an "upper bound" rather than an average for such a class. However, investigating how advanced students integrate formal definitions into their understanding of abstract concepts like subspace can provide teachers and instructional designers with valuable direction as to what kinds of activities students engage in and how those activities can be leveraged into increasingly sophisticated ways of doing mathematics.

Our analysis also demonstrates how investigating a concept's formal definition could become a catalyst for tying together seemingly disparate and potentially conflicting aspects of one's concept image, such as geometric and algebraic intuitions. Our results suggest that emphasizing where definitions can be used to explicitly address this is a productive practice for teachers who want students to better understand the definitions for themselves. For instance, we highlighted issues students have in understanding the ways that parent spaces and subspaces are and are not connected, such as the "nested subspace" conception. Our data shows, however, that students are able to use the definition to deal with and resolve this potential conflict factor. Consider Mark's response below as an example. Early in the interview Mark said that the subspace generated by the vectors in follow-up question (a) would be $\mathbf{R}^3$. The following lines of transcript come after he has been given the formal definition of subspace, and he began, on his own, to reconsider his answer to the question about the three vectors.

Interviewer: They won't in $\mathbf{R}^3$, why do you say they won't in $\mathbf{R}^3$?

Mark: Because these are 4 by 1 matrices; there are four input values. And just because there are three matrices doesn't mean that they span $\mathbf{R}^3$, they actually are in $\mathbf{R}^4$. And the definition, I say that if you're going to add these up, they will not be contained in $\mathbf{R}^3$, and they will be contained in $\mathbf{R}^4$, if you multiply them, yeah.

Interviewer: What's the problem, can you tell me what's the problem?

Mark: I just was so wrong about the definition of subspace before. But I think that now it's up there [points to his head]...I just realized that they have, I think I made the foolish assumption because there were three vectors that they were in $\mathbf{R}^3$...There's actually four coordinates in each vector, which means they're contained in anything larger than, in $\mathbf{R}^4$ or anything larger than that. They actually can't even be contained in $\mathbf{R}^3$. And so, any addition of these two can't be in $\mathbf{R}^3$, because they're going to have four coordinates.

Mark's response came after being given the definition, and he, in light of considering the definition, revised his earlier thinking. By reflecting on the definition, Mark was able to pinpoint where his previous notion of subspace had misled him. His reflection on the definition allowed him to realize and deal with an inconsistency between his earlier notions and his interpretation of the concept definition in a meaningful and appropriate way. The abilities to identify whether personal interpretations are consistent with formal definitions and to revise interpretations if necessary are often considered to be a mark of mathematical sophistication.

Furthermore, the research on using multiple modes of reasoning in linear algebra (Gueudet-Chartier, 2006; Harel & Kaput, 1991; Hillel, 2000; Sierpinska, 2000) have indicated that geometric intuitions about linear algebra can cause difficulties for students as they are making sense of formal aspects of linear algebra. Our data shows that students frequently utilize geometric notions in order to explain what a subspace is for them. While these notions can sometimes be problematic when students begin to work in more than three dimensions, we have shown that when given the definition, students can use their geometric intuitions to make sense of it or to revise their concept images if they do not seem consistent with the definition. Intuitive and geometric notions of subspace may be only problematic if students are unable to identify when their images are inconsistent with the definition.

Teachers want students to have rich and diverse ways of thinking about and reasoning with mathematical ideas; in other words, they want students to form rich concept images that integrate the concept definition in meaningful, productive ways. Zandieh and Rasmussen (2010) discussed the dual role that definitions can play in students' development of mathematical knowledge. Definitions can both describe certain mathematical objects as well as aid in the development of further intuitions about those same mathematical objects and the systems to which they belong. In the case of subspace, the definition can act as a link between concept images of subspace and unexplored notions, such as isomorphism. Being aware of students' common concept images and interpretations of associated definitions has positive pedagogical implications. Not only are teachers made aware of how students think about fundamental ideas with which they have experience, but they can also use this information as the foundation for a meaningful, student-centered development of subsequent mathematical ideas.

Acknowledgements

This material is based upon work supported by the National Science Foundation under grants no. DRL 0634099 and DRL 0634074. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. We are grateful to Chris Rasmussen and Michelle Zandieh for helpful advice during data collection and analysis.

References

Artigue, M. (1992). Cognitive difficulties and teaching practices. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 109-132). Washington, DC: The Mathematical Association of America.

Bernard R. H. (1988). *Research methods in cultural anthropology*. London: Sage Publications.
Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics, 68,* 19-35.
Britton, S., & Henderson, J. (2009). Linear algebra revisited: An attempt to understand students' conceptual difficulties. *International Journal of Mathematical Education in Science and Technology, 40*(7), 963-974.
Carlson, D. (1993). Teaching linear algebra: Must the fog always roll in? *The College Mathematics Journal*, *24*(1), 29-40.
Denzin, N. K. (1978). *The research act: A theoretical introduction to research methods.* New York: McGraw-Hill.
Dorier, J.-L. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational Studies in Mathematics, 29,* 175-197.
Dorier, J.-L., Robert, A., Robinet, J., & Rogalski, M. (2000). The obstacle of formalism in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 85-124). Dordrecht: Kluwer Academic Publisher.
Edwards, B. (1999). Revisiting the notion of concept image/concept definition. In F. Hitt & M. Santos (Eds.), *Proceedings of PME-NA 21* (pp. 205-210). Columbus, OH: ERIC.
Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 221-230). Washington, DC: Mathematical Association of America.
Gueudet-Chartier, G. (2006). Using geometry to teach and learn linear algebra. *Research in Collegiate Mathematics Education VI*, 171-195
Harel, G. (1989). Learning and teaching linear algebra: Difficulties and an alternative approach to visualizing concepts and processes. *Focus on Learning Problems in Mathematics, 11*(2), 139-148.
Harel, G. (2000). Three principles of learning and teaching mathematics: Particular reference to linear algebra – Old and new observations. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 177-190). Dordrecht: Kluwer Academic Publisher.
Harel, G., & Kaput, J. (1991). The role of conceptual entities and their symbols in building advanced mathematical concepts. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 82-94). Dordrecht: Kluwer Academic Publisher.
Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191-207). Dordrecht: Kluwer Academic Publisher.
Larson, C. (2010). *Matrix multiplication: A framework for student thinking.* Manuscript in preparation.
Possani, E., Trigueros, M., Preciado, J. G., & Lozano, M. D. (2010). Use of models in the teaching of linear algebra. *Linear Algebra and its Applications, 432*(8), 2125-2140.
Robert, A., & Robinet, J. (1989). Quelques résultats sur l'apprentissage de l'algèbre linéare en première année de DEUG. *Cahier de Didactique des Mathématiques, n°53*. Paris: IREM de Paris VII.
Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209-246). Dordrecht: Kluwer Academic Publisher.

Stewart, S., & Thomas, M. O. J. (2008). Student learning of basis in linear algebra. In O. Figueras, & Sepúlveda, A. (Eds.), *Proceedings of 32nd PME Conference, and XX PME-NA* (Vol 1, pp. 281-288). Morelia, Michoacán, México: PME.

Strang, G. (1988). *Linear algebra and its applications* (3rd ed.). San Diego, CA: Harcourt Brace Jovanich.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newsbury Park, CA: Sage Publications.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht: Kluwer Academic Publisher.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

Wawro, M. (2009, February). *Task design: Towards promoting a geometric conceptualization of linear transformation and change of basis.* Paper presented at the Twelfth Conference on Research in Undergraduate Mathematics Education, Raleigh, NC.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, IV* (Vol. 8, pp. 103-127). Providence, RI: American Mathematical Society.

Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *Journal of Mathematical Behavior*, *29*(2)*,* 57-75.