Designing Instruction that Builds on Students’ Ways of Reasoning in Linear Algebra

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Part Two: The Magic Carpet Ride
Instructional Sequence
Plan for Part Two of the Workshop

1. Introduce the Magic Carpet Ride (MCR) instructional sequence and our guiding theoretical framework

Day 1
2. Break into small groups and work on Tasks 1 & 2 from the MCR & predict student work
3. Reconvene and discuss

Day 2
4. Break into small groups and work on Tasks 3 & 4 from the MCR & predict student work
5. Reconvene and discuss
The Magic Carpet Ride Problem

• Is a set of four tasks that make use of a realistic problem setting and builds from students’ intuitions toward formal definitions and reasoning

• Promotes a coordinated perspective between algebraic and geometric points of view for vectors and vector equations

• Supports students’ guided reinvention of the concepts of span, linear dependence, and linear independence within the first five days of class

• Has multiple entry points into the four tasks with multiple correct solution strategies for each

• Proves to be a very powerful analogy for students throughout the semester
Background of Research

• Supported by an NSF-funded grant to explore students’ mathematical development in undergraduate courses (PIs: Rasmussen & Zandieh)

• Grant goals, specific to linear algebra:
  • To investigate student conceptions of fundamental ideas in linear algebra
  • To trace students’ intellectual growth
  • To create instructional sequences that support students’ growth

  Magic Carpet Ride problem created to support:
      1) students’ coordination of computation & geometry
      2) students’ understanding of span & linear (in)dependence

• The creation of the MCR Problem was informed by an instructional design theory known as Realistic Mathematics Education
Realistic Mathematics Education

- Instructional design theory originating in the Netherlands (Freudenthal, 1991)

- Is beginning to be adopted for use in undergraduate mathematics courses
  Differential Equations (Rasmussen); Abstract Algebra (Larsen); Linear Algebra (Rasmussen, Zandieh, Wawro, Larson, & Sweeney)

- Instructional sequences are designed to actively engage students in developing the mathematics by starting with students’ current ways of reasoning to build toward more formal, sophisticated mathematics

- Informed by heuristics of guided reinvention and emergent models
Four Levels of Activity
Gravemeijer, 1999; Zandieh & Rasmussen, 2010

**Situational**
Students work toward mathematical goals in an experientially real setting

**Referential**
Involves models-of that refer (implicitly or explicitly) to physical and mental activity in the original task setting

**General**
Involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting

**Formal**
Reasoning reflects emergence of a new mathematical reality; support of prior models-for activities no longer required
Videorecorded each class session and collected student written work

Students had completed at least Calculus I and II, with some having completed Calculus III and/or Discrete Mathematics

Most students were sophomore or juniors and were engineering, mathematics, or computer science majors
Beginning the Semester with Vectors

• Many textbooks begin with systems of linear equations and Gaussian elimination (e.g., Anton, 2010; Lay, 2003; Leon, 2005)

• Alternatively, can begin the course with a focus on vectors, their algebraic and geometric representations, and build towards their properties as sets

• Starting with this instructional sequence has been found to foster:
  ▫ A coordinated perspective between algebraic and geometric points of view for vectors and vector equations
  ▫ The initiation of formal ways of reasoning about ‘objects’ of linear algebra, namely vectors
  ▫ An intellectual need for sophisticated solution techniques and strategies
Magic Carpet Ride Problem

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you a gift. They give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the magic carpet’s movement by the vector \([1, 2]\).
By this we mean that if the magic carpet traveled “forward” for one hour, it would move along a “diagonal” path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

We denote the restriction on the hover board’s movement by the vector \([3, 1]\).
By this we mean that if the hover board traveled “forward” for one hour, it would move along a “diagonal” path that would result in a displacement of 3 miles East and 1 mile North of its starting location.
Task One

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

**TASK:**

*Investigate whether or not you can use the hoverboard & magic carpet to get to Gauss’s cabin.*

- If so, how?
- If not, why is that the case?

As a group, state and explain your answer(s) on the group whiteboard. Use the vector notation for each mode of transportation as part of your explanation. Use a diagram or graphic if it helps illustrate your point(s).
Task One: Sample Student Work

Board 1

Hoverboard // Carpet

- 10 hrs
- 30 ml E
- 10 ml N
- 20 hrs
- 60 E
- 20 N
- 30 hrs
- 90 E
- 30 N

17 hrs 47 hrs 107 E
17E 64 N

Board 2

\[ \begin{align*}
\begin{bmatrix} \frac{3}{1} \\ \frac{1}{2} \end{bmatrix} & = \begin{bmatrix} a \\ b \end{bmatrix} \\
30 & = \frac{90}{30} \\
30 & = \frac{14}{34} \\
\end{align*} \]

\[ \begin{align*}
\frac{90 + 17}{30 + 34} & = \frac{107}{64}
\end{align*} \]
Task One: Sample Student Work

System of Equations: Let $t_1$ =

$3t_1 + 1t_2 = 107$
$1t_1 + 2t_2 = 64$

Elimination:
$6t_1 + 2t_2 = 214$
$t_1 - 2t_2 = -64$
$5t_1 = 150$
Task One: Outcomes from Task

- Students present and class discusses multiple solution strategies
- Teacher tags work and introduces formal notation for:
  - scalar multiplication
  - linear combinations
  - vector equations
  - system of equations
- Coordination of geometric and algebraic views
  - Tip to tail geometry for vector addition
  - Components of vector as horizontal and vertical
  - Scalar/weight as noting change in length of vector or how many vectors in given direction
Task Two

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

- Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot.
- Specify these geometrically and algebraically.
- Include a symbolic representation using vector notation.
- Provide a convincing argument supporting your answer.

Use your group’s whiteboard as a space to write out your progress as you work together.
Instructor: If you would remind me and tell everyone else, how you were thinking about whether or not you could get to Gauss, or whether or not Gauss could hide anywhere.

Debbie: I remember we were discussing a lot...we were drawing graphs. We drew the 2 graphs, the two...the hoverboard and the magic carpet, and we made 2 slopes. And me and Jake were discussing, how in the world can you get to different points on the $x,y$ plane? And we were trying to decide whether or not we could get to any random point. And what I was saying is that, I didn't think you could get to any point in an even amount, without going over or without going under. And then Jake was saying that we probably could, if we used our combination of the different materials, and then being able to backtrack as well.
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*Do the scalars have to be integers?*

*Can the scalars be negative?*
Task Two: Sample Student Work

Board 1

We are restricted to starting at the origin.

The slope of the hoverboard's vector is \( \frac{5}{3} \).

The slope of the carpet's vector is \( \frac{1}{3} \).

Using a line of the form \( y = mx + b \) where \( b \neq 0 \), any \( m \) \( > \frac{2}{3} \) or \( < \frac{1}{3} \) is outside the range of available transportation.

Board 2
Task Two: Sample Student Work

Board 3

Board 4
Task Two: Outcomes from Task

- Students investigate all possible linear combinations by considering “where you can get”
- Students’ work is tagged with the terminology of span of a set of vectors

Definition of span of a set of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \} \) is developed symbolically as the set of all vectors \( \vec{v} \) such that \( \vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_p \vec{v}_p \)
Task Two: Outcomes from Task

- Connect work from Task One to the concept of span
- Continue to work with span with other situational and non-situational examples
Four Levels of Activity

Gravemeijer, 1999; Zandieh & Rasmussen, 2010

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**Formal**
- Reasoning reflects emergence of a new mathematical reality; support of prior models-for activities no longer required
Students explore the definition **span** for sets of vectors. Tasks are asked without specific reference to the original task setting, but students use references to that setting in solving the tasks.

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Students explore different ways of combining travel vectors algebraically and geometrically in the Gauss’s cabin scenario.
Task One and Two: Levels of Activity

- **Situational Activity**
  - Determining how far to travel on each vector to get to Gauss’s cabin
  - Determining which locations you can get to using these two modes of transportation
  - Horizontal Mathematizing occurs in that students interpret the task and their work towards solving this task in terms of various graphical and symbolic forms
  - The instructor helps the students refine their graphical and symbolic notation so that it matches conventional usage

- **Referential Activity**
  - Span as “all possible linear combinations of these vectors” or “all the places you can get to” using these vectors
  - Determining the span of a set of vectors, i.e., determining which locations you can get to using a variety of linear combinations of vectors.
  - Rounding the Corner: Horizontal Mathematizing of working in the task setting starts moving to the Vertical Mathematizing of abstracting to more general situations