The travelling metaphor and its impact on reasoning in our linear algebra course

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Motivation for the study

- Models are student generated ways of organizing their activity with observable and mental tools (Zandieh & Rasmussen, 2010)
  - The Model-of / Model-for Transition
    - The intention of the emergent model heuristic is to create a sequence of tasks in which students first develop *models-of* their mathematical activity, which later become *models-for* more sophisticated mathematical reasoning (Gravemeijer, 1999).

Motivation for the study

- We wanted to know:
  - How did students utilize the model generated in the Magic Carpet Ride Scenario to reason about subsequent tasks and problems?
  - In what ways was the model extended by students to discuss topics that weren't specifically addressed in the original set of tasks?
  - Which aspects of that model might not have been particularly helpful for students in non-related tasks?

The Travelling Metaphor

Metaphor: n. a figure of speech in which a word or phrase is applied to an object or action to which it is not literally applicable.

- **Special usage: A thing regarded as representative or symbolic of something else, especially something abstract.**

New Oxford English Dictionary, 2010

The Travelling Metaphor

In using the MCR tasks, different aspects of a travelling metaphor were identified as being used by students to reason about concepts in linear algebra:

A. A vector is a path in space from a starting point to an ending point.
B. A scalar defines how long you travel on a particular vector
C. A negative scalar is travelling in the opposite direction in space.
D. Linear combinations are paths or routes for getting to specific places in R^2 and R^3.
E. A solution to a vector equation is conceived as whether or not a path to a particular point in space is possible.
F. A set of vectors spanning R^n was conceived as being able to "get everywhere in R^n"
G. Linear dependence was conceived as being able to "get back home" with a given set of vectors.

Research Questions

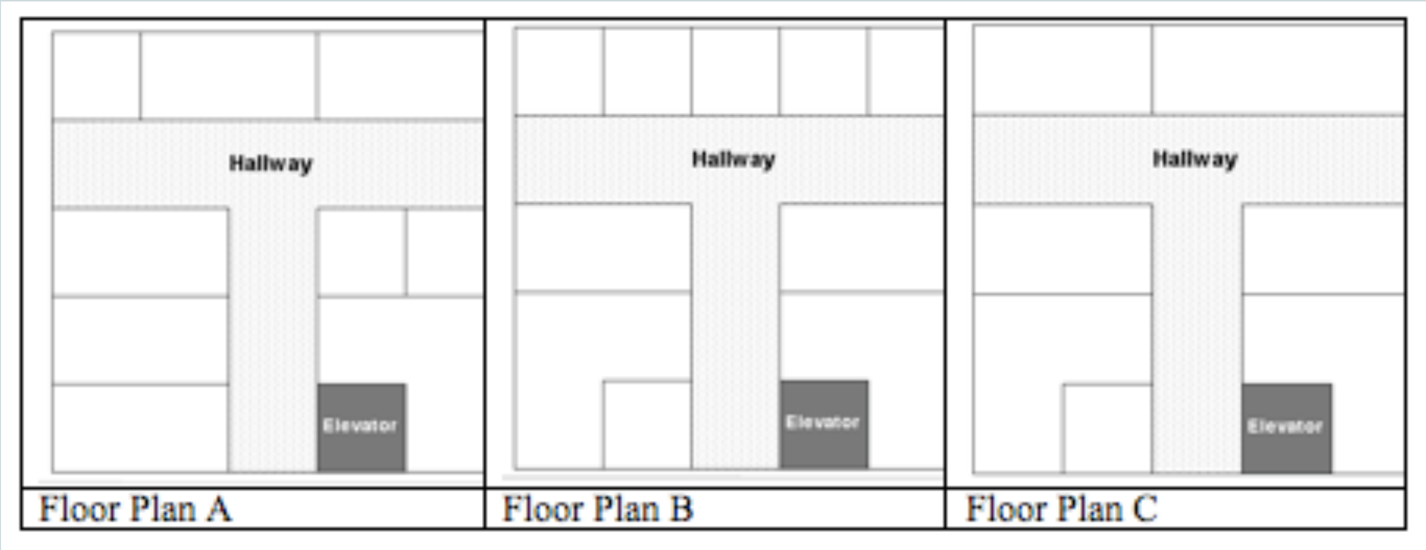
- In what ways did students utilize the travelling metaphor established in the Magic Carpet Ride Sequence to reason about problems or tasks not directly related to movement or the MCR tasks?
  - How did students engage in using the travelling metaphor as it was established in the MCR tasks? (eg. “Getting everywhere” as a way of conceptualizing span.)
  - How did students extend the travelling metaphor in ways that were not discussed during the MCR tasks? (eg. Uniqueness or multiplicity of solutions to a system of equations.)

Methods: Focus Group Interviews

- Six focus groups, with three to four students in each focus group, were conducted in the first three weeks and the last week of the semester.
  - 10 of the 36 students were placed in the focus groups
  - Students were chosen from 6 of the 8 small groups in the class.
  - Students grades in the class varied and exhibited various ability levels with linear algebra.
- Videotapes, audio recordings and transcripts were made of each of the sessions.

The Task: Apartments 'R Us

Apartments 'R Us is a modular construction company that has been contracted to design and build a number of high-rise apartment buildings in cities across the United States. The company uses the following floor plans.



Suppose that you are the contractor and you need to relate how many one, two and three bedroom units there are to a realtor who is selling these apartments? In particular, how many one, two and three bedroom units would you have if a building has 6 floors of Plan A, 11 floors of Plan B, and 17 floors of Plan C?

Additional Tasks

I. Determine how many of each floor plan will be needed to construct a building with 11 three-room units, 23 two-room units, and 17 one-room units. Then produce a general description of how you can figure out how many of each floor plan you would need to achieve any desired combination of three-, two-, and one-room units.

II. Is it possible to create a building with these three floor plans that includes any number of one, two and three bedroom apartments?

III. What if you wanted to find the total number of one, two or three bedroom apartments for a variety of different building plans, how would you go about figuring this out?

IV. Is there more than one group of floor plans that would give you the same number of one, two and three bedroom units for a building? Assume that where each floor plan will be placed in the building is not relevant.

V. A fourth floor plan which has 2 one bedrooms, 1 two bedroom and 4 three bedrooms was added to the original three floor plans. Suppose that the real estate agent wants to have 48 one bedrooms, 74 two bedrooms, and 24 three bedrooms? What set of floor plans would you have to use?

Using the travelling metaphor

- Students in all three focus groups reasoned about whether or not you could get any number of 1,2 or 3 bedroom apartments utilizing language consistent with the Gauss's cabin scenario.
  - Use of the metaphor was primarily restricted to Question II, although in one of the focus groups students also utilized the metaphor significantly in Question IV.
  - Restrictions on movement equate with restrictions on which scalars are allowed for possible linear combinations.
  - "Getting everywhere" is equated with being able to express any vector in R^n as a linear combination of the vectors from the context. Students also related span and being able to "get to" certain locations.

Using the travelling metaphor: Restrictions on travel

- Question II: "Is it possible to create a building with these three floor plans that includes any number of one, two and three bedroom apartments?"
  - Giancarlo: Yeah, he's saying that, because these are linearly independent, you've got to assume you can get anywhere, or you can have any combination. But because of the fact that there is no less than 2 2-bedrooms, you can't have only 1 2-bedroom. So that eliminates the combination right there, judging by the fact that, since we're talking about an amount of something, you can't go backwards. It's not like we're in a vector field where we can go in a negative direction.

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Using the travelling metaphor: Relating span and getting everywhere

Question II: " Is it possible to create a building with these three floor plans that includes any number of one, two and three bedroom apartments?"

- Cassie: If it's linearly independent, then we know that, let's say if you were to graph this, it would be in a space of R^3 in this case. So being linearly independent makes it span all of R^3. And so pretty much any number you plug into the equation, you can get to.

$$6\begin{bmatrix}3\\5\\1\end{bmatrix} + 11\begin{bmatrix}6\\2\\2\end{bmatrix} + 17\begin{bmatrix}1\\3\\3\end{bmatrix} \Rightarrow \begin{bmatrix}18\\30\\6\end{bmatrix} + \begin{bmatrix}66\\22\\22\end{bmatrix} + \begin{bmatrix}17\\51\\51\end{bmatrix} = \begin{bmatrix}101\\103\\79\end{bmatrix}$$

1

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The metaphor drops off

C. Linear combinations are paths or routes for getting to specific places in R^2 and R^3.
   - Two of the groups did not utilize paths or routes for linear combinations or to characterize vectors. Students alternatively discussed the possible combinations of floors and whether particular sets of floors were possible.

G. Linear dependence was conceived as being able to “get back home” using a specific linear combination of vectors.
   - Generally, students showed linear dependence by row reduction, the presence of free variables, the existence of scalar multiples and the number of vectors exceeding the number of dimensions. Students’ activity did not suggest that they were using the “getting back home” portion of the metaphor.
   - Linear independence and dependence was then used to reason about the uniqueness and existence of solutions to systems and their task-associated correlates.

Where the metaphor drops off

- A possible reason that aspect C of the metaphor dropped off in this scenario.
  - The scenario did not lend itself to a geometric interpretation that could be connected to vectors as paths.
- Possible reason that aspect F of the metaphor dropped off.
  - “Getting back home” as a part of the metaphor was created to match the definition.
  - Students utilized equivalent statements to the definition including the presence of scalar multiples, having more vectors than dimensions in the vectors, the presence of a free variable or the fact that a matrix with linearly dependent column vectors will not row-reduce to the identity matrix.

Extending the metaphor: Determining uniqueness of solutions

- Question IV: Is the number of each floor necessary to get a certain number of bedrooms the only set of floor plans that will allow that particular linear combination?
  - Neha: I'm thinking the same thing. I'm thinking like on a graph, I would say assuming you can only use something once, or just one at a time, then you can only have, there's only 1 unique solution.

    Korey: That's true. That makes sense though because even if you had an answer with a negative 4 your first row, go back to the magic carpet, if you first took A and went -4, at the end of the day, you'd probably end up using 4 coming back.

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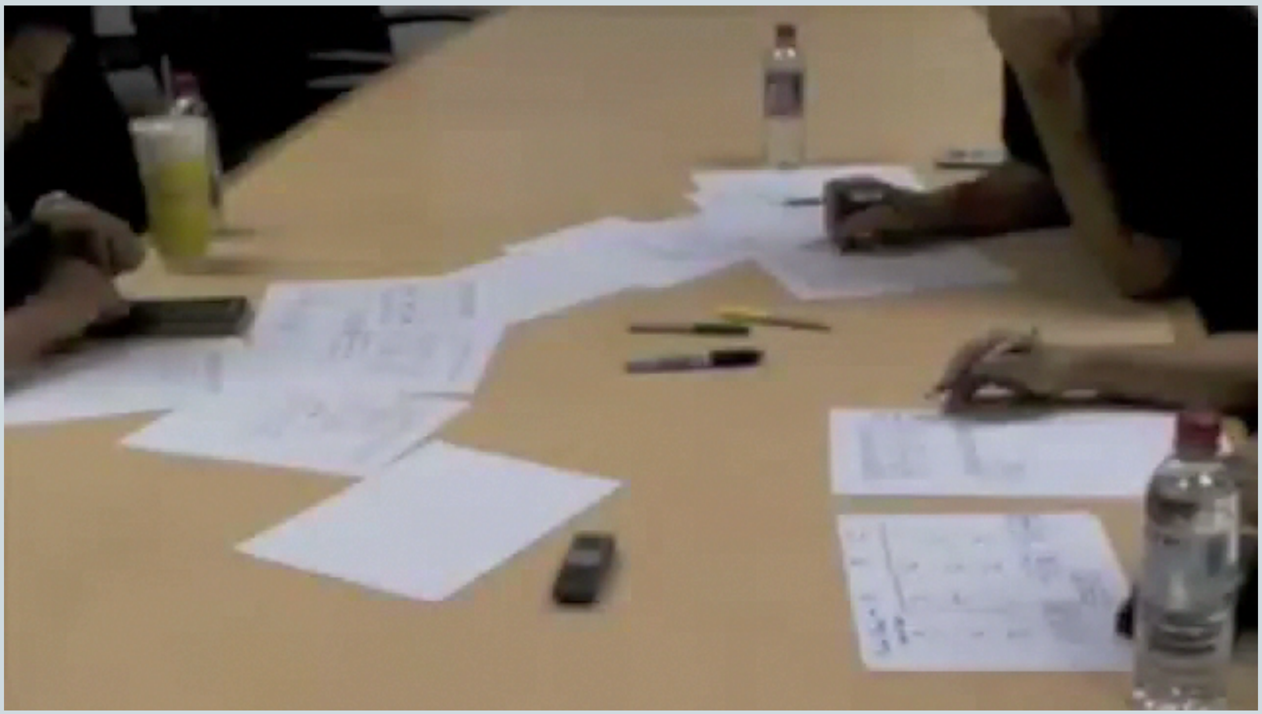
Uniqueness of solutions: Alternative

- Question V: 'A 4th floor plan has 2 1-bedrooms, 1 2-bedrooms and 4 3-bedrooms was added to the original 3 floor plans. Suppose a real estate agent wants to have 48 1-bedrooms, 74 2-bedrooms, and 24 3-bedrooms, what set of floor plans would you have to use?'



Discussion

- Does what we have seen in terms of students use and non-use of the travelling metaphor tell us something about how we might better utilize the travelling metaphor during later portions of the course?
  - The concept of span was consistently related to the idea of "getting everywhere within R^{n}," especially as it related to the range of a linear transformation. However, linear dependence and independence did not get brought back to "getting back home" as consistently when discussing topics that were not related to the Gauss's cabin tasks.
  - The tight connection of "getting back home" to linear dependence was connected primarily to the definition and the development of equivalent statements to the definition. Once students had developed these equivalences, they used them to reason about linear dependence and independence.

Questions

- What are some ways that we can extend the metaphor to include other central ideas in linear algebra?
- Extended metaphors can be very helpful in students' understanding of abstract mathematics. In what ways could they be limiting?