A hypothetical learning trajectory for conceptualizing matrices as linear transformations

Megan Wawro
Virginia Tech

Christine Larson
Vanderbilt University

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Purpose of this Talk

To articulate a hypothetical learning trajectory (HLT) designed to support students' development and elaboration of a transformation view of matrix multiplication.

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- Any transformation T from $\mathbf{R}^n$ to $\mathbf{R}^m$ given in terms of matrix multiplication is a linear transformation when defining $T(\mathbf{v})=A\mathbf{v}$ for a given n x m matrix A
- Consistent with the "matrix acting on a vector" view (Larson, 2011) of the matrix equation $A\mathbf{x} = \mathbf{b}$, in which a matrix acts A on an input vector $\mathbf{x}$ and transforms it into an output vector $\mathbf{b}$

Hypothetical Learning Trajectory (HLT)

- Term first introduced by Simon (1995) to characterize the reflexive nature of designing instruction and considering student learning in the classroom
- Definition (Larson, Zandieh, & Rasmussen, 2008): An HLT is a storyline about teaching and learning that occurs over an extended period of time and includes four interrelated aspects:
    1. Learning goals about student reasoning
    2. The evolution of students' mathematical learning experience
    3. The role of the teacher
    4. A sequence of instructional tasks in which the students engage

Learning Goals of the HLT

- Interpreting a matrix as a mathematical object that transforms input vectors to output vectors
- Interpreting matrix multiplication as the composition of linear transformations
- Developing the idea of an inverse as “undoing” the original transformation
- Coming to view matrices as objects that geometrically transform a space

Setting for Research

- Studied introductory level linear algebra courses in two large public universities in the southwestern United States
- Collected data for analysis by videotaping each class session, collecting student written work, and conducting interviews during 3+ semesters
- Students had completed at least Calculus I and II, with some having completed Calculus III, Discrete Mathematics, or Differential Equations
- Most students were in their 2nd or 3rd year of university and were engineering, mathematics, or computer science majors

First Strand of the HLT

Learning Goal:

Interpreting a matrix as a mathematical object that transforms input vectors to output vectors

Evolution of Students' Learning Experience:

Prior experience with functions serves as a good starting point for this new conceptualization of matrices…

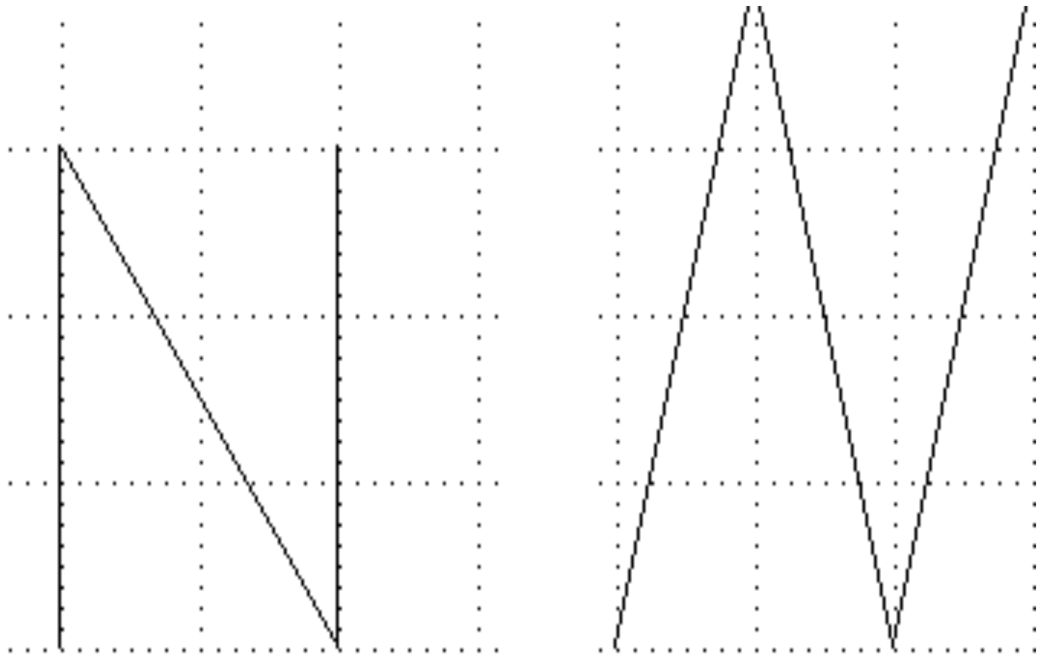
The Role of the Teacher:

- General: Facilitate student interaction & discussion, inquire into student thinking, facilitate mathematical progression of class
- Specific: Define terms, help students connect new notion of 'transformation' to what they know of 'function'…

Main Instructional Task:

General introduction and The Italicizing N Task

The Italicizing N Task

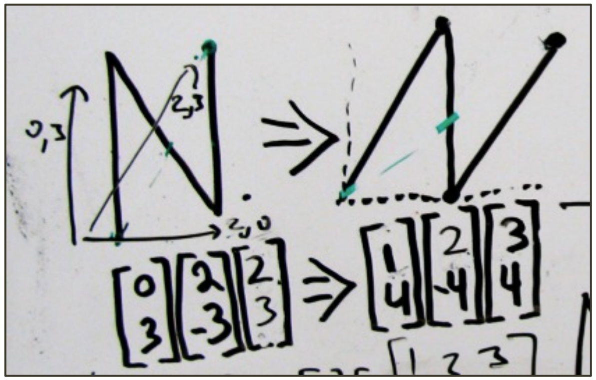


Suppose the "N" on the left is written in regular 12-point font. Find a matrix A that will transform N into the letter on the right, which is written in italics in 16-point font.

A =

The Italicizing N Task

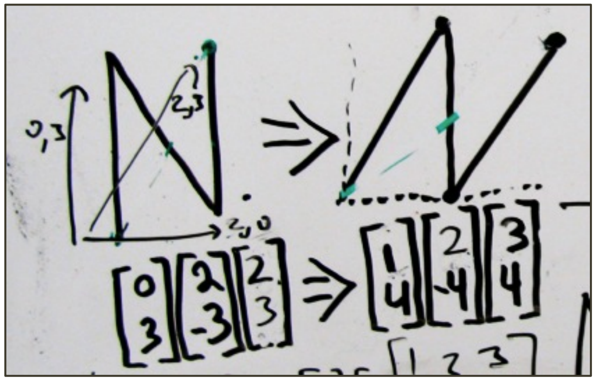
Samples of student work



a) Student work that notates the lines in the “N” as vectors in $\mathbf{R}^2$

The Italicizing N Task

Samples of student work



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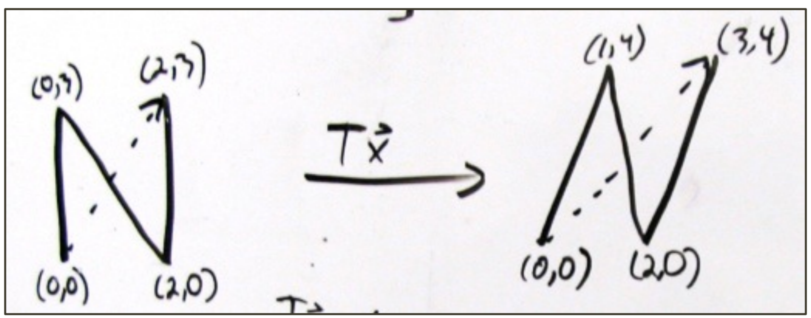
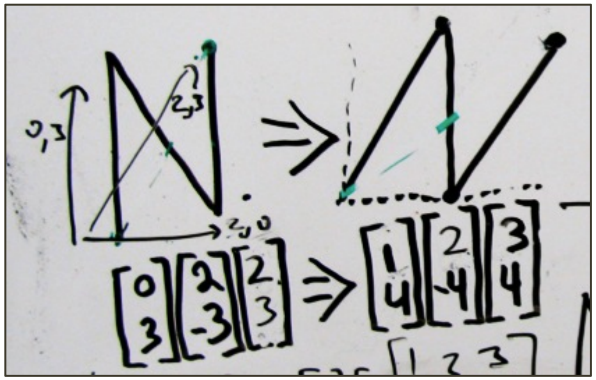


b) Student work that notates corners of the “N” as points on the *x*-*y* plane

The Italicizing N Task

Samples of student work



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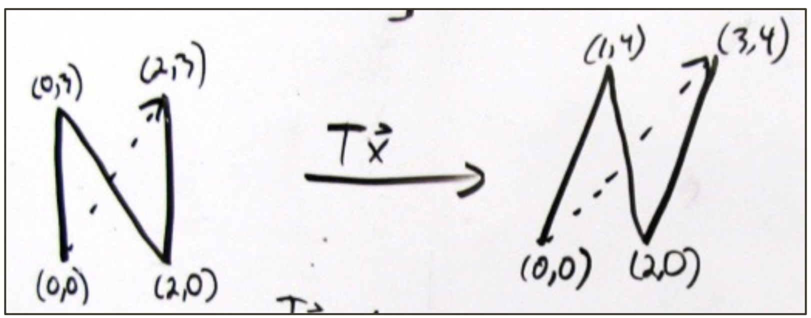

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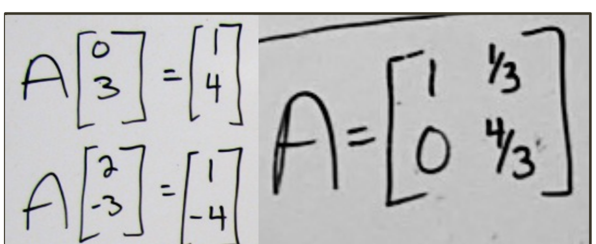


c) Student computation to determine the matrix *A*.

Second Strand of the HLT

Learning Goal:

Interpreting matrix multiplication as the composition of linear transformations

Evolution of Students' Learning Experience:

Students struggle with how to combine transformations in a "sensible" way, have to consider the role of the input/output vectors…

The Role of the Teacher:

Let students grapple with how to "combine" matrices and the order of matrix multiplication, facilitate the connection to composition of functions…

Main Instructional Task:

The Pat and Jamie Task

The Pat and Jamie Task

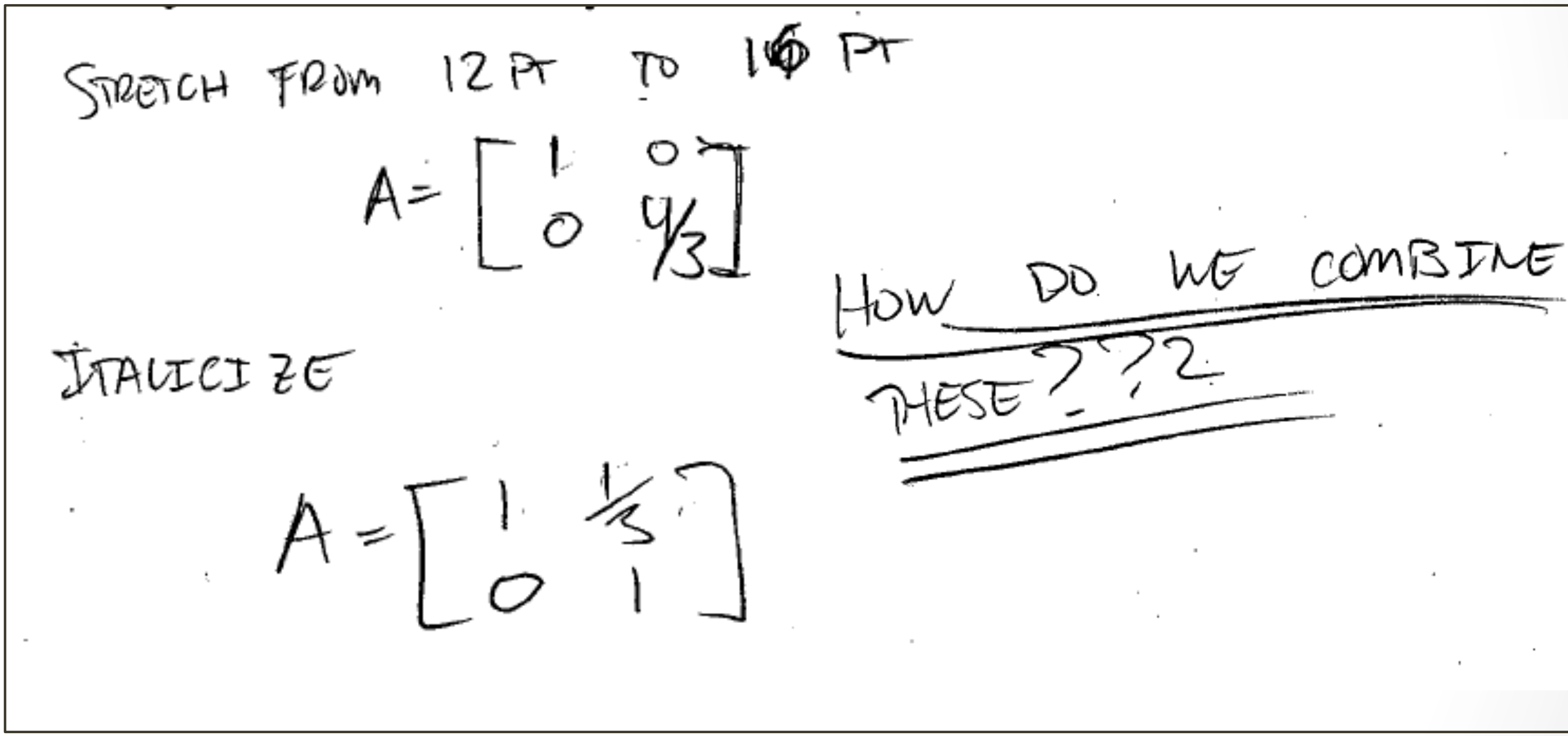
Last semester, two linear algebra students—Pat and Jamie—described their approach to the Italicizing N Task in the following way:

> "In order to find the matrix *A*, we are going to find a matrix that makes the "N" taller (from 12-point to 16- point), find a matrix that italicizes the taller "N," and the combination of those will be the desired matrix *A*."

1. Does their approach seem sensible to you? Why or who not?
2. Do you think their approach allowed them to find a matrix *A*? If so, do you think it was the same matrix *A* we found this semester?
3. Try Pat and Jamie's approach. You should either: a) come up with a matrix *A* by using their approach, or b) explain why this approach does not work.

The Pat and Jamie Task

Samples of student work



Student expresses uncertainty of how to combine the separate transformations to first change from 12 to 16-pt font and then italicize.

The Pat and Jamie Task

Samples of student work

The Pat and Jamie Task

Samples of student work

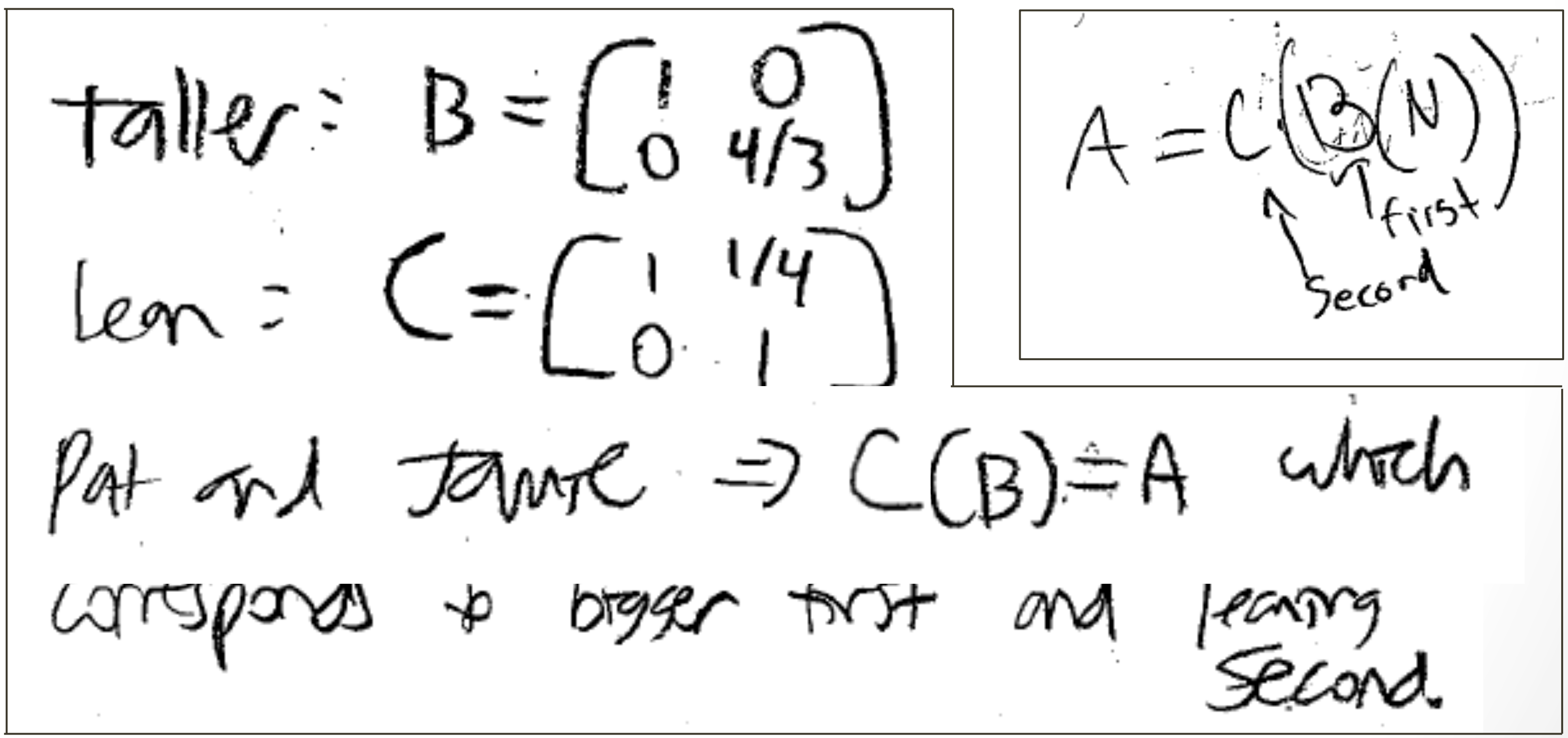
taller: $B = \begin{bmatrix} 1 & 0 \\ 0 & 4/3 \end{bmatrix}$

lean: $C = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}$

Pat and Jamie $\Rightarrow C(B) = A$ which corresponds to bigger first and leaning second.

The Pat and Jamie Task

Samples of student work



Third Strand of the HLT

Learning Goal:

Developing the idea of an inverse as "undoing" the original transformation

Evolution of Students' Learning Experience:

Students pick up from the other 2 tasks and consider going the opposite direction; must consider order of multiplication…

The Role of the Teacher:

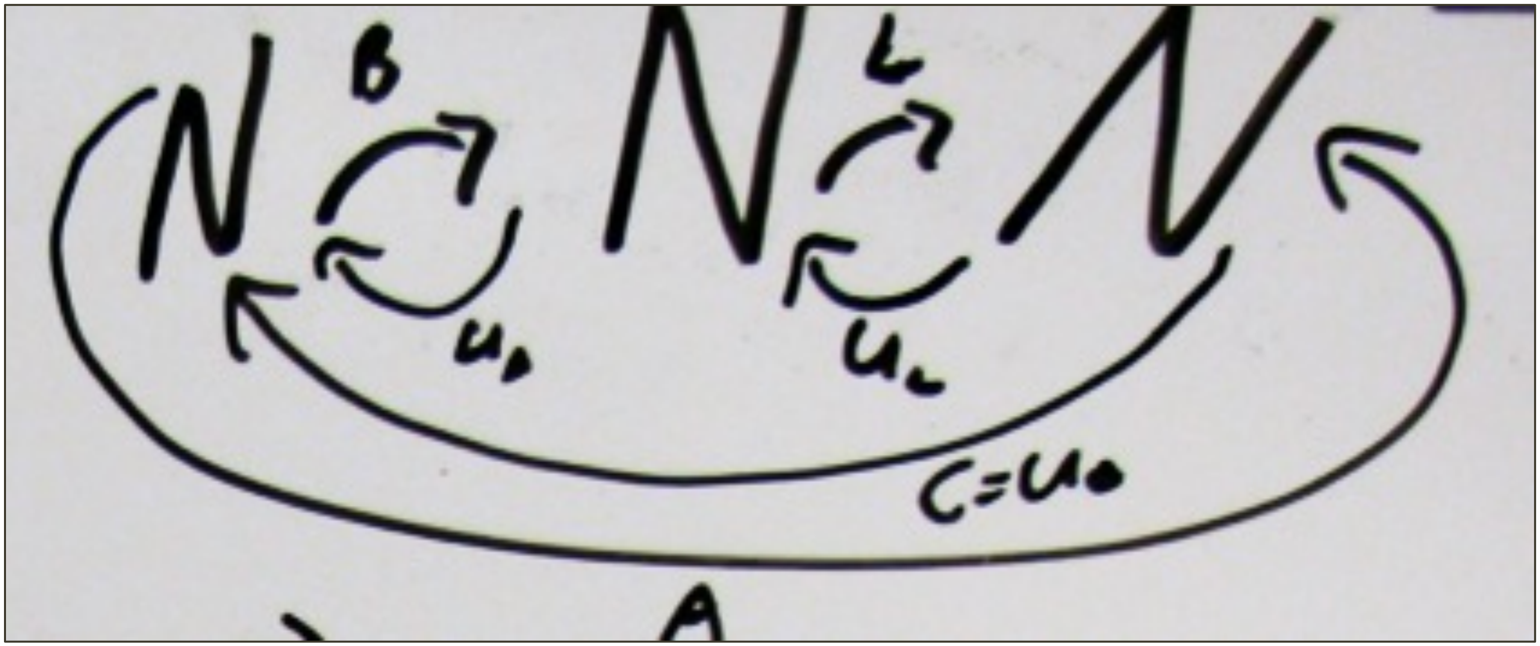
Help develop needed notation, springboard into the formal definition of inverse for both matrices and transformations….

Main Instructional Task:

'Going Back to the N' Task

‘Going Back to the N’ Task

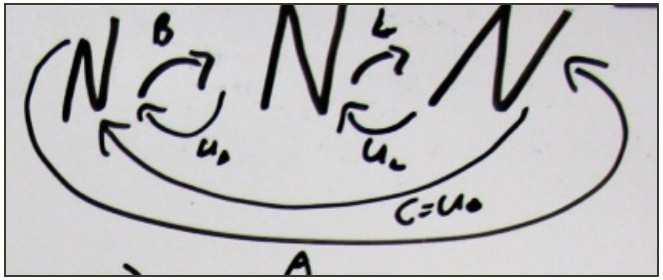
Negotiation of meaning for symbolism



The class negotiates meaning for the matrices U_B, U_L, and U_A, which are the matrices that “undo” the actions of *B*, *L*, and *A*, respectively.

‘Going Back to the N’ Task

Samples of student work



‘Going Back to the N’ Task

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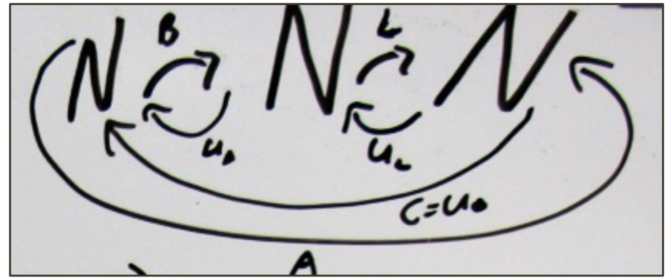
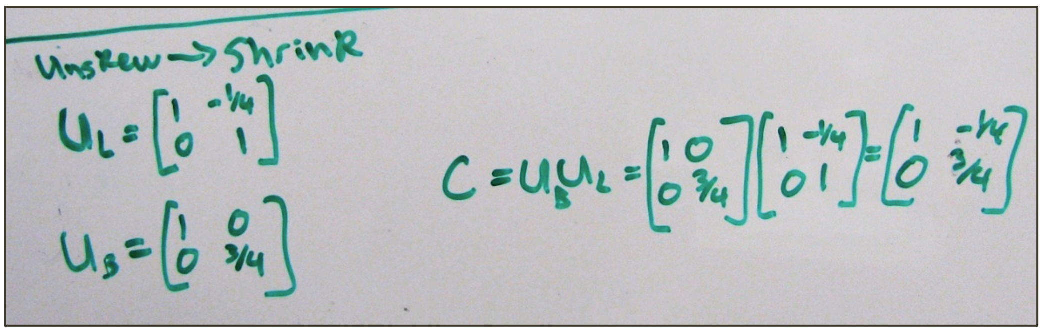
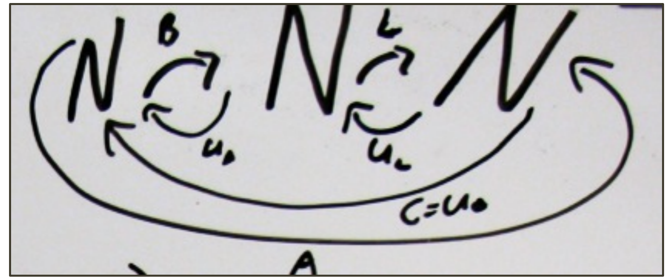


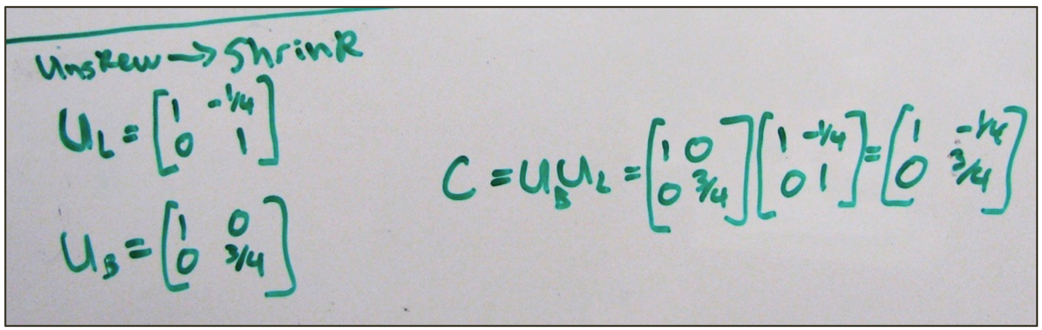


a) student work shows the correct order of computation to find the inverses for Pat and Jamie’s approach

‘Going Back to the N’ Task

Samples of student work





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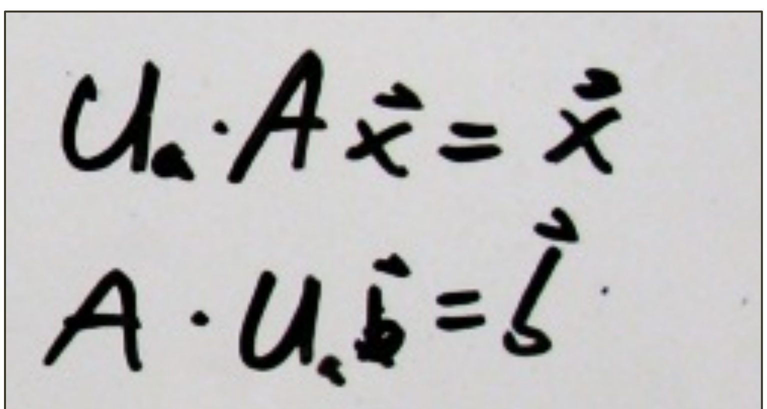


b) A student’s notation showing that *A* composed with its “undoing” matrix U_A in either order leaves the input vector unchanged.

Conclusion

- I summarized a hypothetical learning trajectory (HLT) designed to support students' development and elaboration of a transformation view of matrix multiplication
- It has 4 learning goals:
  - Interpreting a matrix as a mathematical object that transforms input vectors to output vectors
  - Interpreting matrix multiplication as the composition of linear transformations
  - Developing the idea of an inverse as "undoing" the original transformation
  - Coming to view matrices as objects that geometrically transform a space

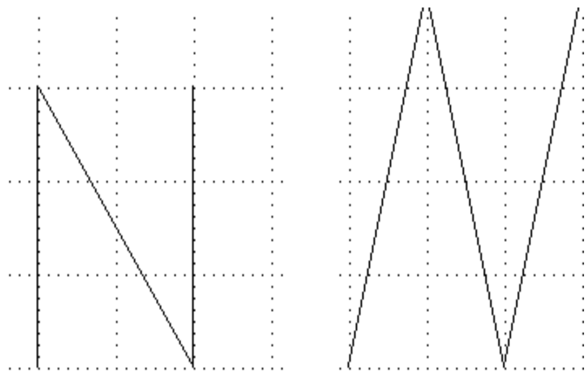
Thank you!

Contact:
Megan Wawro
mwawro@vt.edu

Inverses and the N Task

Regarding the Italicizing N Task, complete the following:

Find a matrix *B* the will transform the letter on the right back into the letter on the left.



1. Find *B* using either your method or one of your classmate's method for finding *A.*
2. Find *B* using Pat and Jamie's method for finding *A*.