An Inquiry-Oriented Approach to Span and Linear Independence: The Case of the Magic Carpet Ride Sequence[1]

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Abstract: In this paper we present an innovative instructional sequence for an introductory linear algebra course that supports students' reinvention of the concepts of span, linear dependence, and linear independence. Referred to as the Magic Carpet Ride sequence, the problems begin with an imaginary scenario that allows students to build rich imagery and formal definitions. The approach begins by focusing on vectors, their algebraic and geometric representations in $\mathbf{R}^2$ and $\mathbf{R}^3$ and their properties as sets. Samples of student work are provided to illustrate the variety of student solutions and how these solutions lead to the creation of formal definitions.

Keywords: Linear algebra, research in undergraduate mathematics education, student thinking, linear independence, span

Running head: The Magic Carpet Ride Sequence

The purpose of this paper is to highlight an instructional innovation that supports students' reinvention of the concepts of span, linear dependence, and linear independence. The instructional sequence and associated data presented are drawn from a three-year research project that investigated the prospects and possibilities for promoting students' understanding in linear algebra. Referred to as the Magic Carpet Ride sequence, the problems begin with an imaginary

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scenario that allows students to build rich imagery and formal definitions, both of which students use to reason throughout the semester. In particular, the instructional sequence leverages students' experiences with travel as the starting point from which to build an intuitive understanding of vectors and vector equations, which then leads to the formal definitions of span, linear dependence, and linear independence. We next share two typical student reactions to the Magic Carpet Ride sequence, followed by a brief explanation of the guiding theoretical and methodological background for this work. The main section of the paper then details and illustrates the instructional sequence.

As illustrated in the following excerpt from an individual problem-solving interview conducted with a student, Jerry, midway through the semester, the Magic Carpet Ride sequence offers students rich imagery to make sense of formal definitions.

Interviewer: Different people think about concepts in mathematics differently. I want to know, how you think about linear independence?

Jerry: Linear independence. Really, it goes way back to the first Magic Carpet problem. I think it's actually the first thing that enabled me to grasp it. Because I did take the class once before, and it was just definitions. I never would have thought of it as being, you have a set of vectors, can you get back to the same point? That never occurred. It's all abstract if you don't have that analogy.

Similar comments were expressed by many of the other students in the classes we observed. What Jerry meant by "can you get back the same point?" is explained in the subsequent sections. For now, our point is that the Magic Carpet Ride sequence offers students powerful imagery that enables them to make sense of formal definitions. This is particularly important because previous research has demonstrated that formal definitions are often a stumbling block for students [4, 5, 21].

The Magic Carpet Ride sequence also provides students a way to competently talk about the mathematics to people who are not familiar with linear algebra. For instance, at the end of Kaemon's problem solving interview he, without prompting, reflected on his experiences in previous mathematics classes and then contrasted that with his experience in linear algebra.

Kaemon: [Referring to his previous math classes] Usually I just focus on the mechanics and how to do the problem. I just plot through them and I don't really care what they mean…So this class, it makes me think of a way I could explain math to someone that doesn't know anything. That's how I feel. Now I have the words to describe it to someone who has [knows] nothing about linear algebra. Since a lot of our problems are in the context of real-world type of examples, like the magic carpet ...There's ways to say, explain linear algebra in common, not really common language, but language that's easier than just the book, I'd say.

The reflections by Jerry and Kaemon are not unique. Many other students in the classes we observed expressed similar sentiments.

Guiding Theoretical and Methodological Background

Our efforts to create innovative instructional sequences in which students learn mathematics with meaning has been guided by the instructional design theory of Realistic Mathematics Education (RME) [6]. A central tenet of RME is that mathematics is first and foremost a human activity, as opposed to being a predetermined collection of truths. This "implies a change in perspective from decomposing ready-made expert knowledge as the starting point for design to imagining students elaborating, refining, and adjusting their current ways of knowing" [7, p. 106]. Consistent with this change in perspective, the course structure is based on two levels of inquiry. On one level, students learn mathematics though inquiry by participating in mathematical discussions, explaining their thinking, and solving novel problems. On a second level, instructors inquire into their students' mathematical thinking to make decisions and guide classroom activity [17]. The complex and active role of the instructor includes facilitating student discussions, modeling appropriate argumentation behavior, and utilizing student ideas and justifications to move forward the mathematical development of the class.

RME-inspired instruction is being adapted to the undergraduate curriculum in a number of content areas, including differential equations [17], geometry [29], abstract algebra [12, 13], and calculus [16, 24]. Perhaps the most mature of this work is in differential equations, where comparison studies with students taught in a traditional lecture-style have been conducted. These studies reveal that students in the RME-based classrooms scored comparably on tasks that measured procedural fluency but significantly better on tasks that measured conceptual understanding of the material [18]. Moreover, students in RME-based differential equations courses retained their knowledge better than students in the traditionally taught classes [11].

Methodologically, our work is grounded in what is typically referred to as "design research" [10]. Design research is a way to engineer and analyze innovative learning environments. Our design research interventions involve iterations of instructional design and planning, ongoing analysis of classroom events, and retrospective analysis of all data sources [2]. To date we have conducted three semester-long interventions at two different universities with three different teacher-researchers, involving over 100 students. Students typically had completed at least two semesters of calculus, with some students having completed a third semester of calculus or a discrete mathematics course. All students had previously encountered the concept of vector in their previous courses. Most of the students were in their second or third year of university and had chosen engineering, mathematics, or computer science as their major course of study. We collected data for analysis by videotaping each class session, collecting student written work, and conducting interviews with students throughout the semester. All data presented in this report are drawn from data collection efforts from the spring 2010 semester.

The Magic Carpet Ride instructional sequence is one product of our design research work. Other products include instructional sequences on linear transformation, eigen theory, and change of basis, all of which follow a similar format to that of the Magic Carpet Ride. Qualitative analyses of student reasoning indicates that our RME inspired approach holds considerable promise for promoting student understanding of key ideas in linear algebra [9, 19, 27].

The Magic Carpet Ride Instructional Sequence

The Magic Carpet Ride instructional sequence begins on the first day of class, prior to any formal instruction, and consists of four main problems. These four problems typically take five to six class sessions to complete. Students work in small groups of three to five students per group. Small group work is alternated with whole class discussions in which students explain their tentative progress, listen to and attempt to make sense other groups' progress, and finally come to justified conclusions on the problems. We also provide each group with a whiteboard to allow them to facilitate working together and to publically share their thinking with the rest of the class. The Magic Carpet Ride instructional sequence is compatible with the aforementioned inquiry-oriented setting in that most tasks are sufficiently open-ended to allow for multiple solution strategies and representations. Furthermore, because none of the tasks are trivial for students, they are challenged with debating each other's answers and the validity of various solution strategies. We contend that these aspects are fundamental to the efficacy of the instructional sequence in supporting students' reinvention of the mathematical concepts.

Problem One: Investigating Vectors and their Properties

In problem one, students investigate whether it is possible to reach a certain location—the location where Old Man Gauss lives—with two different modes of transportation (see Figure 1). The goals of problem one are to (a) have students present and discuss multiple solution strategies; (b) have the teacher use student work as a starting point for introducing formal notation and language for scalar multiplication, linear combinations, vector equations, and system of equations; and (c) coordinate geometric and algebraic views of the problem and its solution. One of the modes of transportation given in problem one is a magic carpet. Its movement, when ridden forward for a single hour, is denoted by the vector $\begin{bmatrix}1\\2\end{bmatrix}$ to indicate motion along a "diagonal" path resulting from displacement of 1 mile East and 2 miles North of its starting location. The other mode of transportation, a hoverboard, is defined similarly along the vector $\begin{bmatrix}3\\1\end{bmatrix}$. The problem as it is given to students is shown in Figure 1.

<< Insert "*Figure 1*: Problem One of the Magic Carpet Ride sequence" here >>

In order to develop a shared understanding of the notation to be used and the assumptions inherent to the problem context, the class discusses the idea that, if one were to ride the magic carpet forward for, say three hours, one would end up 3 miles East and 6 miles North of the starting point, and that the journey could be denoted as either $3\begin{bmatrix}1\\2\end{bmatrix}$ or $\begin{bmatrix}3\\6\end{bmatrix}$. The class also discusses what it would mean to move "backward" on the modes of transportation and for fractions of an hour, as well as how these movements could be represented with vector notation.

We show the work of four of the eight groups from the Spring 2010 data collection in Figure 2 to illustrate the variety of ideas students brought to bear on the task. It was *not* immediately obvious to students that Old Man Gauss can be reached (namely, by riding the magic carpet forward for 17 hours and the hover board forward for 30 hours), so there was a substantial amount of mathematical work for students to do just with this initial task.

<< Insert "*Figure 2*. Samples of student solutions to the first problem" here >>

These examples were selected to illustrate the variety of approaches students used. In our experience, students' strategies tend to split fairly evenly among three broad approaches: (a) guess and check, (b) system of equations first, and (c) vector equation first. The guess and check groups (such as groups 8 and 5 in Figure 2) tend to think more in terms of the location to be reached, preferring a guess-and-check accumulation strategy that is represented either geometrically with vectors (like group 5) or numerically (like group 8). The vector equation groups (such as group 3) often begin with a vector equation, which they then rewrite as a system of equations to be solved. The systems of equations groups (such as group 2) choose to begin with a system of equations outright, which they solve and reinterpret with vectors.

The variety of approaches indicate that students have varying levels of fluency in using vector notation, in representing systems of equations, and in coordinating their algebraic and geometric understandings. By allowing each group to share its approach to the problem, students have the opportunity to see the problem in multiple ways and make new connections between how they are thinking and other possible approaches. For example, group 8's work suggests that its members were not conceiving of time as a weight in the sense of its multiplicative relationship with the vectors – however, they would likely say that their approach was similar to that of group 5. After seeing group 5 present, members of group 8 are likely to be more aware that the amount of time each mode of transportation is ridden can be conceived of as a weight for each vector.

These diverse student approaches allow the instructor to bring out and make explicit important ideas and connections. For example, during Spring 2010, the instructor used student work to make explicit how to transition between vector equations and systems of equations. The instructor also helped the class appreciate the connection between students' vector equation and various geometric representations. For instance, some groups drew the vectors as right triangles to emphasize their component parts. Other groups elected a tip-to-tail depiction. Still other groups chose to include a depiction of the resultant vector of the tip-to-tail method. All of these depictions are common and important ways of graphically representing vectors and their addition, and it is the instructor who is in the position to make these topics of discussion. Another topic of discussion the instructor may bring forth from this task is how time is depicted in students' graphical representations of the solution. Conceptually this is significant because the interpretation of time as concatenation or stretching of vectors may not be immediately obvious to many students. As our brief discussion of this problem makes clear, the instructor plays an essential role in bringing out and helping to formalize and connect student thinking to the conventions of the mathematical community. While on the surface the problem for students is to figure out if they can use the two modes of transportation to reach Old Man Gauss, the use of student work to bring out the mathematical richness and significance is the responsibility of the instructor.

Problem Two: Reinventing the Notion of Span

The second problem in the instructional sequence asks students to determine whether there is any location where Old Man Gauss can hide so that they would be unable to reach him using the same two modes of transportation from the previous problem. The goal of problem two is to help students develop the notion of span in a two-dimensional setting before formalizing the concept with a definition. The problem as it is given to students is shown in Figure 3.

<< Insert “*Figure 3.* Problem Two of the Magic Carpet Ride sequence” here >>

From the same data set previously discussed, after working on problem two in small groups, the task of determining if Old Man Gauss could hide became reinterpreted as determining if you could “get everywhere” with the two modes of transformation provided, and there was disagreement as to whether it was in fact possible to “get anywhere.” In part, we attribute this disagreement to varying student interpretations of “backward” travel, despite the fact that initial discussion of the task explicitly addressed this issue. Some groups did not assume you could travel backwards, and others interpreted traveling backwards as traveling backwards in time. Still others assumed that time always moved forward, so the sign of the scalar coefficient of each vector indicated whether the corresponding mode of transportation was being ridden forwards or in reverse. These differences turned out to be productive because they forced students to wrestle, in productive ways, with their interpretations of vector addition, scaling, and linear combinations of vectors and the variety of ways they might appear in geometric depictions. Figure 4 provides four samples of student work that represent the range of student approaches to this problem.

<< Insert "*Figure 4*. Samples of student solutions to the second problem" here >>

At the heart of students' difficulties in this task was the issue of developing a coherent geometric interpretation for linear combination of vectors with all possible cases for sign combinations of scalar coefficients. As students worked on this task, they began to develop the ability to conceptualize movement in the plane using combinations of vectors. In looking at the definition of span, it may seem obvious that a task intended to help students develop an intuitive understanding of span should require students to investigate the idea of linear combinations in depth. However, we see here that it is a non-trivial task for students to explore and develop a concept image [25] for span in which all possible linear combinations of vectors are conceptualized in a coordinated way.

Class discussion of this task set the stage for the instructor to introduce the formal definition of span as follows: The span of a set of vectors is all possible linear combinations of those vectors, or in other words, all places you could reach with those vectors. Furthermore, any vector that can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p$ for some real numbers $c_1, c_2, ... c_p$ is in the span of $\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_p\}$.

Problem Three: Reinventing Linear Independence/Dependence

In problem three, students are asked to determine if, using three given vectors that represent modes of transportation in a three-dimensional world, they can take a journey that starts and ends at home (i.e., the origin). The problem as it is given to students is shown in Figure 5. The restrictions that are placed on the movement of these modes of transportation are that the vectors could only be used once for a fixed amount of time represented by the scalars c_1, c_2, and c_3. The purpose of problem three is to provide an opportunity for students to develop geometric imagery for linear dependence and linear independence that can be leveraged in the development of the formal definitions of these concepts.

<< Insert "*Figure 5.* Problem Three in the Magic Carpet Ride sequence" here >>

As with the other tasks, students first work on this problem in their groups, and the class conversation alternates between this and "sharing out" in whole class regarding groups' progress and difficulties with the task. During the Spring 2010 semester, initial progress on this problem was made when the class established that a trip that began and ended at home could be represented by the vector equation $c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}4\\1\\6\end{bmatrix}+c_3\begin{bmatrix}6\\3\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. By formulating the problem in this way, students were able to connect their algebraic activity of previous tasks to their work on this problem. Two student solutions are shown in Figure 6.

<< Insert "*Figure 6*. Samples of student solutions to the third problem" here >>

Whole class discussion of the various approaches offered insight into the ways students were thinking about linear combination of the three vectors. For example, one student visualized that a solution for a journey that began and ended at home would form a triangle in 3-space with the three given vectors, describing the triangle as "riding out on two of the vectors and using the third one to get back home." The language of "getting back home" came to represent for these students the movement along the vectors and how to combine the vectors so that the journey returned to the origin. The instructor labeled the ability to "get back home" with the term linearly dependent, and she subsequently introduced the formal definition of linear dependence as follows: Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$, if there exists a solution to the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$ where not all $c_1, c_2, \ldots, c_p$ are zero, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a linearly dependent set.

The instructor then asked students if there was a way that a rider could "get back home" using the magic carpet and hoverboard vectors from problem one, which became known as "the Gauss's Cabin vectors." In this discussion, students did not always appeal to the formal definition of linear dependence. Instead, they referenced their intuitive understanding of "getting back home" as how they knew that the vectors were not linearly dependent, highlighting the importance of the Gauss's cabin scenario in making sense of linear dependence and independence. To further solidify the connections between "getting back home" and the formal definition, one of the students demonstrated that by placing the vector equation into a system of equations, the scalars c_1 and c_2 would be forced to be zero. The instructor then tagged the two vectors in the Gauss's Cabin scenario as linearly independent and introduced the formal definition of linear independence as follows: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is a linearly independent set if the only solution to the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$ is if all $c_1, c_2, \ldots, c_p$ are zero.

In order to give students the opportunity to work with the formal definitions, the instructor supplied the students with several sets of vectors to determine whether or not they were linearly independent. One such set contained the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. By inspection, one student noted that you could ride out on the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to the point (4,4,4) and then ride back on $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. Thus, because there was a way to leave and return home using two of the three vectors, the set of three vectors is linearly dependent. How exactly this underlying imagery fit the definition was then made a topic of investigation by the instructor. Specifically, she had the students decide how it was that the three vectors fit the definition for linear dependence or independence. The students concluded that the scalar multiple for the first would be four and for the last would be negative one, and they could use zero as the scalar for the middle vector. In the process of discussing how this particular solution fit the definition, students had to deal what it meant for a set of scalars to be a solution, and also what the definition meant when it said a non-zero solution. From their small-group and whole class discussions, members of the class came to the conclusion that if a particular scalar was zero that the set of vectors was linearly dependent, as long as other scalars in the solution were not. This discussion highlighted the role that the scalars played in determining independence or dependence, thus making the connection between the underlying imagery of "getting back home" and the definition of linear dependence stronger.

Students often encounter obstacles with the notions of linear independence and dependence because of the difficulty in interpreting the formal definitions and using formal systems [3, 22]. In the Magic Carpet Ride sequence, students' work provides them with rich geometric and algebraic imagery for linear independence/dependence, imagery that is strongly connected to the formal definitions. Jerry's reflection quoted at the beginning of the paper exemplifies this. Students are given the opportunity to come to conclusions about what is meant by a solution and a non-zero solution and what it meant to require all c_i to be zero, allowing students to create a meaningful connection to the formal definition.

Problem Four: Generalizing

Whereas in the previous task students were given sets of vectors and had to determine if the vectors were linearly independent or linearly dependent, problem four asks students to create their own sets of vectors for four different conditions (see Figure 7). Students are also asked to make generalizations, however tentative, regarding linear independence and dependence. The goals of problem four are (a) to guide students to develop generalizations and supporting justifications regarding linear independence and dependence for any given set of vectors, shifting away from a dependence on the Magic Carpet Ride scenario, and (b) to develop a need for more sophisticated solution techniques, such as Gaussian elimination.

<< Insert "*Figure 7:* Problem 4 in the Magic Carpet Ride sequence" here >>

During the Spring 2010 semester, discussion of student work began with the instructor circulating about the room to review each group's whiteboard to determine how best to ground the ensuing discussion in students' examples. The most interesting and most debated aspect of problem four was when students were asked to create an example of a set of three vectors in $\mathbf{R}^2$ that formed a linearly independent set. In Figure 8, two groups' answers—which were not in agreement—are shown.

<< Insert "*Figure 8.* Group 2's and Group 4's whiteboards, highlighting discrepancy in answers regarding an example of a set of 3 vectors in $\mathbf{R}^2$ that is linearly independent" here >>

Group 2 suggested that $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}$ was a linearly independent set of vectors, whereas Group 4 suggested that there was no possible solution for that particular scenario. When the instructor called attention to this, it led to a class-wide, student-led debate about the discrepancy in those answers. This debate, as well as ensuing conversation about the rest of the examples, realized our two goals: making generalizations and developing an intellectual need for sophisticated solution techniques.

Making generalizations. The small groups worked to fill out the chart in Figure 7 and develop various generalizations about linear independence and dependence. The degree to which the students were confident about the conjectured generalizations varied among the students as well as among the specific generalizations. The four generalizations listed below, which occurred during the Spring 2010 semester, represent typical responses.

1. If you have a set of vectors in $\mathbf{R}^n$ where two of the vectors are multiples of each other, then the set is linearly dependent.
2. If any vector in the set can be written as a linear combination of the other vectors, then the set is linearly dependent.
3. If the zero vector is included in a set of vectors, then the set is linearly dependent.
4. If a set of vectors in $\mathbf{R}^n$ contains more than n vectors, then the set is linearly dependent.

In light of the "getting back to home" scenario of the Magic Carpet Ride problem, the first two conjectures were easily understood and justified by students. For instance, a common approach for creating examples of linearly dependent sets was to use vectors that were scalar multiples of each other. During class, students justified this approach geometrically through the context of the Magic Carpet Ride problem ("If I ride the vector <2, 2> out, I can ride the other vector <1, 1> to get back home, so those two vectors are a linearly dependent set") as well as in terms of the definition of linear dependence ("there exists nonzero solutions to the vector equation $c_1\begin{bmatrix}2\\2\end{bmatrix}+c_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, such as $c_1=1,\ c_2=-2$"). The third and fourth conjectures, however, required much more debate and justification. In order to facilitate these important conversations, the instructor showed the two discrepant responses shown in Figure 8 to the students. After they established that not both groups could be correct, the instructor asked for volunteers to explain their thinking for either argument.

Instructor: I heard the table in the back say they were confident in the top one [Group 2's answer]. Can I have a volunteer from your table to come up and tell us why you're confident?

Gabe: Basically, this breaks out a problem of number theory. If you believe zero is a number, you go with us. That's my campaign.

Justin: [Raises hand] Can I rebuttal?

Instructor: I don't understand the original one yet. Can somebody tell me what you said there?

Gabe: Zero vector, if <0, 0> can be a vector, ours is correct.

Nate: Yeah, but at the same time, if you say <0, 0> is a vector, by the linearly dependent definition, we say that you can use it if one of them, if one constant is not equal to zero, so we set c_1 and c_2 as zeroes and the c_3 is set to anything, and it's still a linearly dependent set.

Gabe: So the definition says that they all have to be zero.

Justin: And the last one can be any number you want.

Robert: Look, if you have a zero vector, then every, every set of vectors will be dependent.

Justin: Exactly!

Robert: If you put any coefficient in front of that <0, 0>, and it would be not all equal to zero.

Abraham: Including 2,564 or whatever that is up there.

...

Robert: If you put, if you want to say that all of them can be zero to get linear independence, then if you put any coefficient in front of that <0, 0> and say it's okay to have any coefficient in front of that, then it's not ever going to be linearly independent. Because you won't have a zero *c* in front of that zero vector.

Justin: Can I say that again?

Instructor: You can go ahead, you can say it again.

Justin: So what he's saying is, for it to be linearly independent, all the *c*'s must be zero, they have to be zero. So if we chose zero for the first two, that's great, fine. But for the zero vector, we can chose any number, 8,462 if we want, and it's still going to be a solution that's zero.

This transcript highlights how students referred to the formal definition of linear dependence in order to justify why any set containing the zero vector must be a linearly dependent set. There is a growing body of mathematics education research that documents the challenges students face in using formal definitions and reasons for these challenges [5, 26, 28]. For example, it is well documented that students often use their concept image rather than the concept definition [25] to make assertions and defend their claims. While doing so at times is useful and necessary, students also need experiences and opportunities to defend their assertions based on formal definitions. It is noteworthy that the Magic Carpet Ride sequence affords students with opportunities to develop and use both the imagery of the situation as well as formal definitions to justify claims. Another noteworthy point regarding this problem was the class's interest in how the zero vector could be interpreted as a mode of transportation. The way the class thought about this was as a stationary bicycle. Specifically, you can ride the stationary bicycle (i.e., the zero vector) for as long as you want (i.e., have a nonzero scalar), and you will never leave home.

As the discussion continued, the students realized that just because Group 2 was incorrect did not necessarily mean that Group 4 was correct in conjecturing that there was no set of three vectors in $\mathbf{R}^2$ that was linearly independent. An additional argument was required to justify this assertion. Justin, a member of Group 4, supplied the following justification in class:

Justin: So we're still in $\mathbf{R}^2$. So basically, let's just start with any random vector, let's call it that one. Now after we have one vector down, there's only basically two situations we could have. We can either have a vector that is parallel with this one, either another multiple or going the other way or whatever. Or we can have one that is not parallel, it doesn't have to be perpendicular, it can be anywhere. But it's either parallel or not. So if it's parallel, we already said that if we have two vectors that are parallel, we have a, they're dependent. But when we did our magic carpet-hoverboard, we had two that weren't parallel, and we said the span of any two that aren't parallel, is all of $\mathbf{R}^2$. So if we have two that aren't parallel, and we can get anywhere in $\mathbf{R}^2$, no matter where we throw in our third vector, we can get there with a combo of these two and make it back on that third one. So there can't be any solution, so there's no, as long as we have three vectors in $\mathbf{R}^2$, it has to be linearly dependent. Does that make sense, any questions?

Justin continued his argument and explained how the exact same argument applies for $\mathbf{R}^n$, using $\mathbf{R}^3$ as an example of how the argument generalizes to other dimensions.

Developing an intellectual need for sophisticated solution techniques. The second major goal of problem four is the development of an intellectual need [8] for an efficient computational strategy to achieve the desired goal. It is common for students to assume that if by inspection they cannot immediately "see" the dependence relation among a set of vectors in $\mathbf{R}^n$ for n greater than or equal to three, then the set must be linearly independent. The instructor is in the position to draw attention to this, stating that sometimes it is difficult to be certain from inspection alone with or not a set of vectors is linearly dependent. Thus, they needed a more rigorous solution strategy. Consider the following transcript as an example:

Instructor: One other aspect about the worksheet [shown in Figure 7] was that when we got to the certain part of the table, it's just kind of hard to check. Some of you were tricky about the way you picked some of your vectors. Some of you just picked random ones, which is totally fine. But it wasn't always easy to check whether they're linearly independent or dependent or not. So it's where we want to go today. So the goal would be after a couple of days to try to figure out a way to check more rigorously if things are in the span. And to check more rigorously or computationally if things are linearly independent or dependent, so to develop a computational method to check sets for linear independence or dependence, and if given vectors are in the span.

Based on this need to check rigorously and efficiently for properties of sets of vectors, the instructor subsequently introduced new terminology and solution techniques. She reminded the students of the terms *vector equation* and *system of equations*, and she introduced for the first time the term *matrix* as an array of numbers whose columns are the vectors in question. Making the connection to vector equations, she defined the operation $A\mathbf{x}$ as the linear combination where the entries of $\mathbf{x}$ are the weights for the respective columns of A. Furthermore, she defined the equation $A\mathbf{x} = \mathbf{b}$ in such a way that the vector $\mathbf{b}$ is the result of A times $\mathbf{x}$, or the result obtained when the entries of the vector $\mathbf{x}$ weight the column vectors of matrix A to form a linear combination. She also introduced *augmented matrices* and *equivalent systems* to set up the method of row-reduction and Gaussian elimination. Rather than simply being the next chapter or the next unit for study, Gaussian elimination germinated from a genuine need within this classroom: a need to be certain that a set of vectors is or is not linear independent, and a need for an efficient computational strategy for solving large systems of equations.

Conclusion

To address how the instructional approach described here compares to other approaches, we examined a number of popular university linear algebra textbooks in the United States. In contrast to our starting point with vectors, we found that most texts begin with systems of linear equations and Gaussian elimination[1] (e.g., [1, 15, 20]). One possible reason for beginning the course in this manner is that students have prior experience in solving systems of equations. As early as high school algebra, for example, students solve systems of two equations with two unknowns. One might even say that by the time students reach linear algebra, systems of equations are part of students' intuitive background. We strongly agree that curriculum should begin with content that has an intuitive basis for students. However, motivating and developing the formal notions of span and linear (in)dependence by starting with systems of equations is not

[1] The most recent text by Strang [23] includes a section within the first chapter that introduces linear independence and dependence through two examples in $\mathbf{R}^3$, and Gaussian elimination appears in Chapter 2.

easy to do. Indeed, students often have difficulty with what seems to them to be an abrupt change in perspective from systems of equations to vector equations [14]. What we propose in this paper is choosing a different intuitive starting point from which to build and structure an introductory linear algebra course.

Our approach to an introductory linear algebra course begins by focusing on vectors, their algebraic and geometric representations (in $\mathbf{R}^2$ and $\mathbf{R}^3$), and their properties as sets. In the body of this paper we demonstrated that starting with vectors and vector equations fosters the development of formal ways of reasoning in a way that connects to students' current ways of reasoning and motivates the need for sophisticated solution techniques and strategies such as Gaussian elimination. As the previous sections illustrated, the Magic Carpet Ride instructional sequence begins the semester in such a way consistent with this shift in initial focus. Students are asked to investigate and solve problems related to these foci, and these explorations are carried out in $\mathbf{R}^2$ and $\mathbf{R}^3$ within the problem context created by the Magic Carpet Ride setting. From these explorations, solutions, and generalizations—which students present to each other in order to explain and justify their thinking—students essentially reinvent the conceptual underpinnings of span and linear (in)dependence. The instructor then tags these new concepts with conventional terminology and definitions. As the class begins to investigate these properties with sets of vectors in $\mathbf{R}^2$, $\mathbf{R}^3$ and $\mathbf{R}^4$ that are not tied to the Magic Carpet Ride context, they quickly realize the inefficiency and unwieldiness of guess-and-check, substitution, and elimination as solution strategies. That is, during their own process of developing understanding of the objects of linear algebra (namely, investigating properties of sets of vectors), the class sees the need for powerful and efficient representational forms and solution techniques, such as matrix equations and

Gaussian elimination. The move to matrices then sets the stage for further work with matrix equations, linear transformations, and eigen-theory.

A next stage in this work is to set up comparison studies that investigate the effectiveness of our RME-based approach to linear algebra instruction. Here, by effective we mean that the students would score comparably or better on items that measured computational/procedural fluency as well as on conceptual understanding of the material. Another issue for future study involves how the MCR sequence supports or constrains students' understanding of key ideas regarding abstract vector spaces. We also plan to develop special sessions at national mathematics conferences to share this ongoing work and to further develop, assess, and refine the instructional sequence.

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THE MAGIC CARPET RIDE PROBLEM

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

PROBLEM ONE: THE MAIDEN VOYAGE

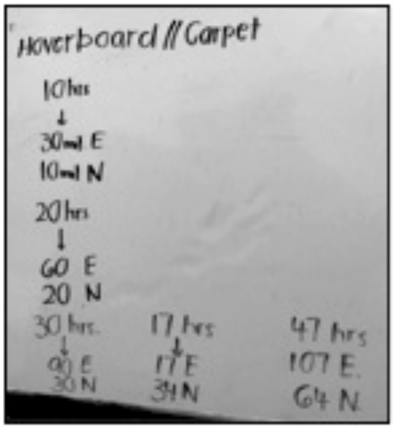
Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

TASK:

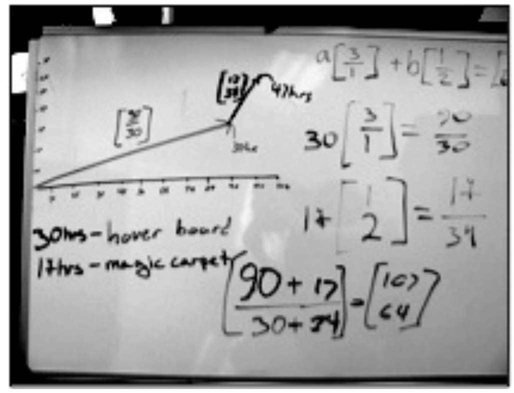
Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin.

If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case? Use the vector notation for each mode of transportation as part of your explanation. Use a diagram or graphic to help illustrate your point(s).

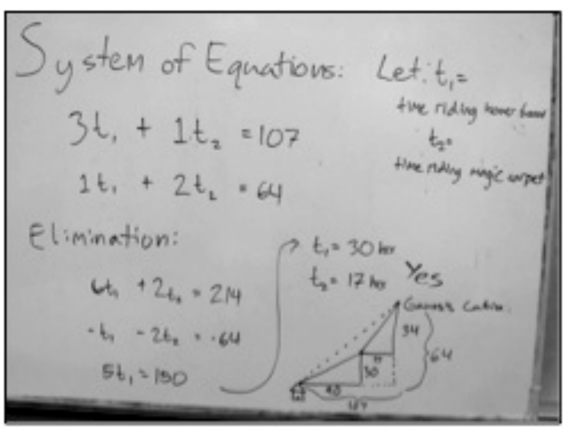
Figure 1. Problem One of the Magic Carpet Ride sequence



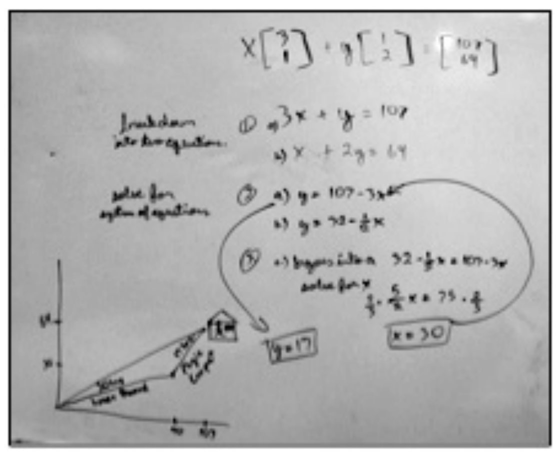
Group 8's "Guess and check" method: This group began by using the hoverboard 10 hours at a time. After iterating this 10-hour journey three times and recognizing that this got them to a location of 90 miles E and 30 units N, they determined that riding the magic carpet for 17 additional hours would allow them to arrive at their desired destination of 107 miles E and 64 miles N.



Group 5's "Guess and check via vector weighting" method: This group began by writing a vector equation (but with 'fraction-like' notation). They determined the weights that each vector would need so the East and North components would sum to 107 and 64, respectively. They included a graphical representation where the weighted vectors are depicted tip-to-tail.



Group 2's "Systems of equations" method: This group began by defining variables (t_1 and t_2) for the amount of time spent on each mode of transportation and representing the problem using a system of equations. They used elimination to solve the system and drew a diagram to indicate the journey corresponding to their solution. In their diagram they used right triangles to illustrate the component parts of each vector that made up their journey to Gauss's cabin and included a dashed line to depict the resultant vector.



Group 3's "Vector equation becomes a system" method: This group began by representing the problem using a vector equation. They rewrote the vector equation as a system of equations, which they solved. Finally, they depicted the journey corresponding to their solution with vectors drawn tip-to-tail. They also included a depiction of the resultant vector.

Figure 2. Samples of student solutions to the first problem

THE MAGIC CARPET RIDE: PROBLEM TWO

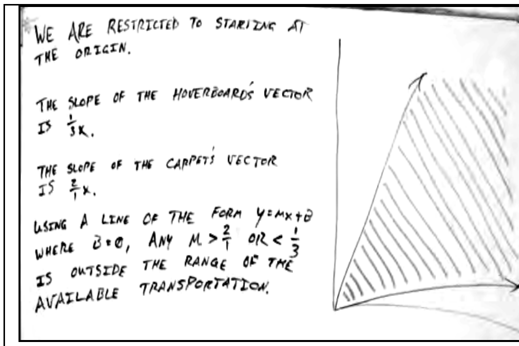
SCENARIO TWO: HIDE-AND-SEEK

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Use your group's whiteboard as a space to write out your work as you work together on this problem.

Figure 3. Problem Two of the Magic Carpet Ride sequence

<table>
<tr>
<td>
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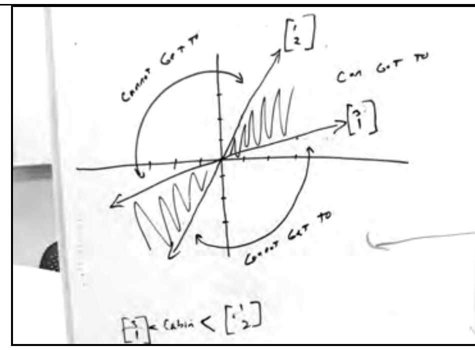
<td>
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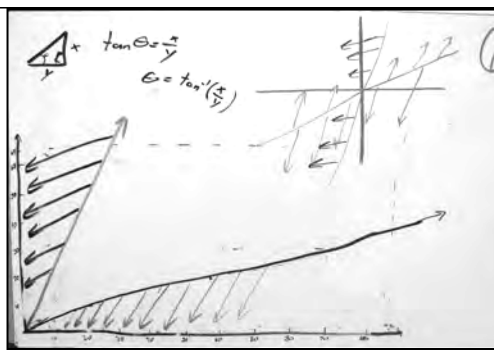
<tr>
<td>Group 2's "Cone" method: This group argued that the only points that could be reached were the ones that lie "between" the cone traced out by the extensions of the two transportation vectors. Their argument was framed in terms of the slopes of the lines corresponding to the vectors when drawn from the origin.</td>
<td>Group 1's: "Double cone" method: This group interpreted the sign of the scalar as an indication of whether they were moving forward or backward in time. So, in their interpretation, either both modes of transportation had to move forward (cone in first quadrant) or both had to move backward (cone in third quadrant).</td>
</tr>
<tr>
<td>
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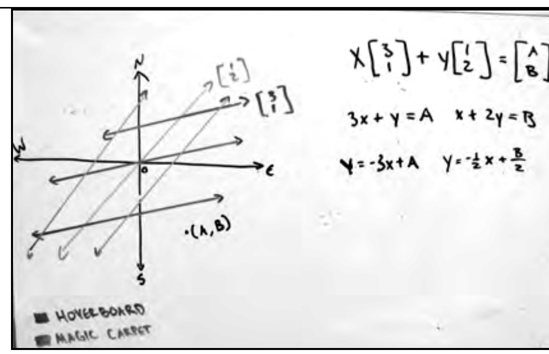
<td>
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<tr>
<td>Group 6"s "Zig zag" method: This group argued that you can reach any point on the plane by taking into consideration the ability to ride any given mode of transportation backwards. They explained that the portions of the graph that Group 1 deemed unreachable were accessible when considering that you can travel in the negative direction. For example, to travel to a point located in the 2nd quadrant, you travel in the positive direction a set distance with the magic carpet then travel in the negative direction with the hover board. Because each vector can be extended to any desired length through scalar multiplication, every point on the graph is reachable.</td>
<td>Group 7's "Grid" method: While this group did not formally present their solution to the class, their idea of "gridding" came up in whole class discussion and led to a student asking, "Can we use any scalar to slide to any point on the graph?" This question was resolved by exploring when a vector equation equivalent to that shown at the top of this board had a solution.</td>
</tr>
</table>

Figure 4. Samples of student solutions to the second problem

THE CARPET RIDE PROBLEM: DAY THREE

SCENARIO THREE: GETTING BACK HOME

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$.

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\mathbf{v}_1$, c_2 on $\mathbf{v}_2$, c_3 on $\mathbf{v}_3$). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home OR explain why it is not possible to do so.

Figure 5. Problem Three in the Magic Carpet Ride sequence

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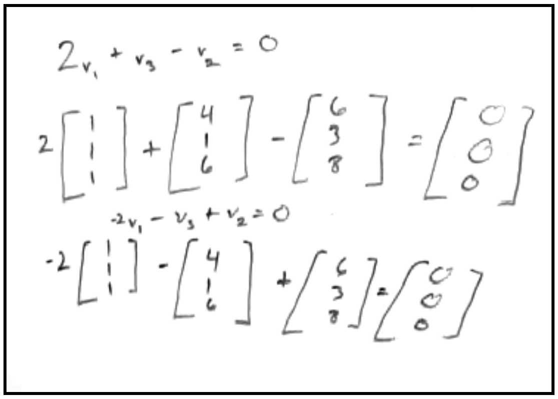
<td>
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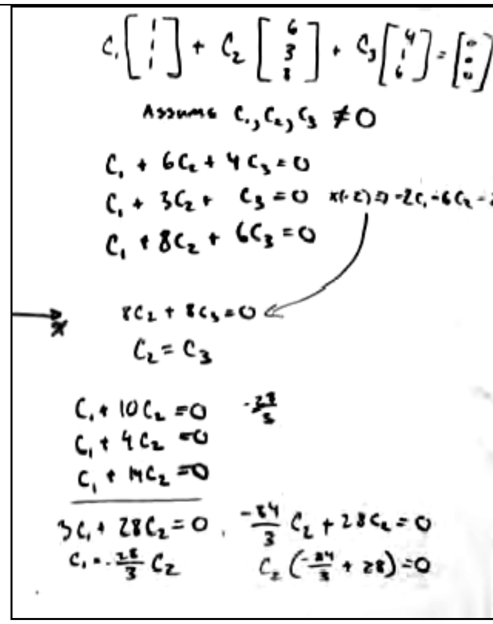
<td>
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<td>Group 5's "Guess and check" approach: The students in this group solved by inspection that the zero vector could be obtained by taking a linear combination of the three given vectors. A group member said, "Pretty much just an observation that we noticed, that if you multiply the 1st vector by 2 and add it into the 3rd vector, you get the same amount as the 2nd vector. So that's why I picked that equation."</td>
<td>Group 7's "Systems" approach: This group translated the vector equation into a system of equations in order to solve for the necessary scalars. The board shown contains some mathematical errors and does not display a complete solution.</td>
</tr>
</table>

Figure 6. Samples of student solutions to the third problem

LINEAR INDEPENDENCE AND DEPENDENCE: CREATING EXAMPLES

Fill in the following chart with the requested sets of vectors.

	Linearly dependent set	Linearly independent set
A set of 2 vectors in $\mathbf{R}^2$		
A set of 3 vectors in $\mathbf{R}^2$		
A set of 2 vectors in $\mathbf{R}^3$		
A set of 3 vectors in $\mathbf{R}^3$		
A set of 4 vectors in $\mathbf{R}^3$		

Figure 7: Problem 4 in the Magic Carpet Ride sequence

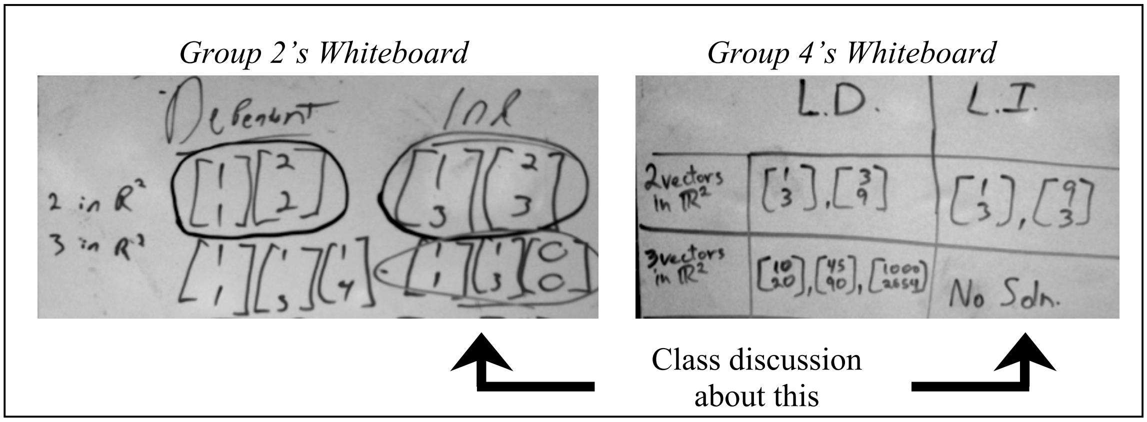

Figure 8. Group 2’s and Group 4’s whiteboards, highlighting discrepancy in answers regarding an example of a set of 3 vectors in $\mathbf{R}^2$ that is linearly independent